Investigating and Improving Student Understanding of Superposition in Introductory and Quantum Physics Courses

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Abstract

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This dissertation presents results from research and curriculum development related to student understanding of the principle of superposition in introductory optics and upper-division quantum mechanics courses. The focus is on the extent to which students are able to relate the mathematical formalism used in physics to real-world phenomena. In particular, the investigation involves three topics: (1) student ability to determine quantum probabilities for discrete and continuous cases and to recognize similarities between the procedures, (2) student ability to recognize the measurable effects of relative phase in superposition states, and (3) student ability to apply superposition to the interference of classical waves. In each case, specific difficulties that students encounter when reasoning about superposition are presented, together with a discussion of how curriculum was developed to promote student learning and to address common errors. The process through which the curriculum is developed and its impact on student understanding of superposition are discussed explicitly.

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DEDICATION

For those who believe in passion and perseverance

Chapter 1. Introduction and Overview

The principle of superposition is widely applied in physics, mathematics, and engineering. For linear systems, the net effect due to two or more stimuli is the sum of the effect due to each individual stimulus. For example, the electric field due to a set of charges can be considered as the vector sum of the electric field due to each individual charge. Other common applications are wave interference, Fourier analysis for circuits, quantum superposition, etc.

In quantum mechanics, superposition is one of the most fundamental principles. Quantum states can be superimposed just like classical waves. Any quantum state can be expressed as a linear combination of basis states. However, quantum mechanics is distinct from classical mechanics, in large part due to the fact that it is probabilistic in nature, while classical mechanics is deterministic. That said, one can predict the behavior of a classical system with certainty, but one can only predict the behavior of a quantum system with some probability.

The Physics Education Research (PER) community, including the Physics Education Group at the University of Washington (UW PEG), has been putting effort into research on student understanding of quantum mechanics [1-35]. Some groups focus on student conceptual understanding and qualitative reasoning in various topics [1-29]. Others center on student problem-solving and metacognitive skills [30-35]. One overarching conclusion is that many students lack a functional understanding of key concepts and principles in quantum mechanics after instruction. While examining student conceptual understanding of a variety topics in quantum mechanics, we (UW PEG) have found that many common difficulties students encounter may be due, at least in part, to failure to reason critically about superposition. We thus decided to probe in detail student ability to apply superposition in quantum mechanics, which we believe can provide further insights into student understanding of many other topics that build on this fundamental principle. The research findings have guided us to develop and revise curriculum that is intended to improve student understanding and reasoning skills in upper-division quantum mechanics courses.

Wave theory is applied in both classical and quantum mechanics. It is typical that students have learned interference of classical waves in introductory physics courses before they take upper-division quantum mechanics courses. The literature shows that student understanding of similar concepts or ideas can "interfere" mutually, resulting in decreased performance [36,37]. We assumed that student understanding of interference of classical waves can influence (positively or negatively) their ability to reason about quantum interference. Investigating introductory students' ability to apply superposition to interference informs us about quantum students' prior state of reasoning ability, but also allows us to identify the patterns of lines of reasoning used by introductory and quantum students. Although there exists prior research on investigating and improving introductory students' understanding of interference [38-40], the study discussed in this dissertation focuses specifically on the extent to which students are able to apply superposition and how it can be improved.

This dissertation describes an investigation into student understanding of superposition in introductory and quantum physics courses, as well as assessment of curriculum that has been developed based on our findings in this investigation. Three major areas are involved in this study: (1) student ability to determine quantum probabilities for discrete and continuous cases, (2) student ability to recognize the measureable effects of relative phase in superposition states, and (3) introductory-level students' ability to use superposition to reason about interference of classical waves. Regardless of the different physics scenarios, superposition is one of the underlying principles of these three topics. The first topic requires a functional understanding of quantum states that can be and are often represented by vectors in Hilbert space in which superposition is applied. The second and third topics involve relating phase differences between individual basis states (or individual waves) to the results of superposition.

The main body of this dissertation is divided into three parts based on the three research areas mentioned above. Each part starts with relevant prior research, followed by the investigation into student understanding. Then we describe curriculum that has been developed based on our research findings. Lastly, we present results from curriculum assessment.

1.1 Overview of Prior Research

In introductory-level physics courses, research has been conducted to probe student ability to use superposition in various contexts, including vectors, electrostatics, and waves [38-45]. Barniol and Zavala, for example, investigated the effects of different contexts on student understanding of vectors [41]. Singh identified student difficulties in applying superposition to Gauss's law [42-45]. Ambrose et al. found that many students fail to use the path length (or phase) difference to make qualitative predictions about interference patterns [38]. Wosilait et al. showed that a sequence of tutorials successfully addressed various difficulties students have in applying a wave model to the interference and diffraction of light [40]. In chapter 6, we will discuss in detail how prior research on investigating and improving student understanding of interference sheds light on our study.

Research on learning quantum mechanics has involved various perspectives, such as student problem-solving and mathematics proficiency [30,33-35], conceptual understanding [1-29], epistemological beliefs [31,32], etc. Among all these studies, conceptual understanding has been the primary focus. The specific findings that are relevant to this study will be discussed in each subsequent part of this dissertation. In this section, we summarize some overarching results.

Student conceptual understanding of quantum mechanics has been documented across a wide variety of topics that include, but are not limited to, the wave function [2,3,27], time dependence [6-9,26], quantum measurement [1-4,7], tunneling [21,28], Dirac notation [20,22,23,29,33]. One of the overall findings is that many students lack a functional understanding of some fundamental concepts in quantum mechanics after traditional lecture instruction. It is common that students do not distinguish between quantum concepts and classical concepts. Zhu and Singh, for example, found that many students confuse Hilbert space with three-dimensional lab space [10]. Wittmann et al. pointed out that students often state that particles lose energy while tunneling [21]. Moreover, many students have difficulty with determining probabilities and with qualitative predictions about measurements and time dependence [6-8]. They do not often distinguish between the eigenstates for different observables or recognize the effect of measurement on a quantum system.

The findings from this body of research have informed development of instructional materials that are intended to supplement traditional lecture instruction. The PhET simulations [46] developed by the University of Colorado – Boulder, for instance, cover a variety of topics in quantum mechanics (as well as in science and mathematics). These animated and interactive simulations allow students to explore how different parameters can change the results. The Quantum Interactive Learning Tutorials [47] from the University of Pittsburgh make use of computer-based visualization tools. The UW PEG has also been developing and assessing tutorials that are intended to improve student understanding of quantum mechanics. Similar to *Physics by Inquiry* [48] and *Tutorials in Introductory Physics* [49] developed by UW PEG, the tutorials for upper-division quantum mechanics use guided inquiry as a general instructional approach.

The PER community has also been putting efforts into developing and refining instruments [3,24,27] to assess student learning outcomes. Similar to the Force Concept Inventory [50] for Newtonian mechanics, these instruments often make use of sets of multiple-choice questions to assess student understanding of fundamental concepts in quantum mechanics. Many have been tested at both undergraduate and graduate levels.

1.2 Context for Research

In this section, we introduce the context in which this study has been carried out. We start with student populations and courses. We then describe the relevant prior tutorials developed by UW PEG.

1.2.1 Student Populations and Courses

The students involved in this study mostly come from the UW, a public research university with an enrollment of more than 40,000 students. Students' average high school GPA is between 3.64 and 3.93 [51]. A variety of courses were covered in this research, ranging from introductory calculus-based sequence to upper-division physics courses.

The introductory calculus-based sequence comprises PHYS 121 (Mechanics), PHYS 122 (Electricity, circuits, and magnetism), and PHYS 123 (Waves, optics, modern physics/heat). The textbooks that have been used are Tipler and Mosca [52] (before Winter 2016) and Mazur [53] (since Spring 2016). There are usually between one and four classes for each course in a given quarter. Enrollment for each class is typically about two hundred students. Most of these students are engineering or science majors. Physics majors comprise less than 5% of the total enrollment.

Each of these courses has three components: lecture, laboratory, and tutorial (instead of traditional recitation) sections. Each week, there are three 50-minute lectures, a 50-minute tutorial section, and a 110-minute lab. Lecture sections usually make use of interactive methods such as clickers. Lab sections have a primary goal of reinforcing the content presented in lectures.

Each tutorial contains three parts: a pretest, an in-class worksheet, and homework. The pretests are short online quizzes that are intended to start students to think about the concepts that will be covered in worksheets. Another purpose is to inform instructors and TAs about what students have mastered and what conceptual and reasoning difficulties they have even after lecture instruction. Students receive credit for completion but not for correctness. They also receive credit for attending the tutorial section. Students are encouraged to work in small groups (3 to 5 students each group with a total of about 24 students) and discuss their thought processes. Teaching assistants (TAs) are taught to help students articulate their ideas and reasoning by asking guiding questions. The homework reinforces or extends the material covered in worksheets. Similar to the worksheets, homework emphasizes explanations, which counts on a large portion of homework grades.

Most of the research at the introductory level discussed in this dissertation has been carried out in PHYS 123, the last part of the introductory calculus-based sequence mentioned above. Students have seen examples applying superposition in the contexts of forces and electric and magnetic fields in PHYS 121 and PHYS 122. The relevant topics in PHYS 123 that use superposition are waves and optics.

The other two courses in which this research has been conducted are PHYS 225 (Introduction to quantum mechanics) and PHYS 324 (Quantum mechanics). PHYS 225 is a sophomore-level quantum physics course that covers the first six chapters of the McIntyre [54] textbook. It has three 50-minute lectures each week. Interactive methods such as clickers and tutorials have been implemented by instructors since 2014. This course is required for all students who will major or minor in physics. Usually there are more than a hundred students enrolled. Prerequisite includes PHYS 227, the first of a two-quarter sequence on mathematical methods in physics, in which students have learned complex numbers and Fourier series.

PHYS 324 is the first of a two-quarter sequence junior-level quantum mechanics. It has two 80-minute lectures and a 50-minute tutorial every week. The tutorial section is similar to the introductory tutorial section and uses the instructional materials developed by UW PEG. The students enrolled, varying from 70 to 110, are all physics majors. Together with PHYS 325, the second part of the sequence on quantum mechanics, they cover all the chapters in the Griffiths textbook [55]. Typically, students have completed PHYS 225 and the two-quarter sequence on mathematical methods in physics before they take PHYS 324. PHYS 324 is required for almost all students in physics, while PHYS 325 is optional and is usually only taken by students who plan to apply to graduate school.

1.2.2 Relevant Prior Tutorials Developed by UW PEG

The UW PEG has been developing tutorials that are intended to improve student conceptual understanding and scientific reasoning skills, which traditional instruction often fails to achieve. The tutorials facilitate student learning through guided inquiry. Critical concepts and essential skills are targeted to support students to develop a functional understanding of important but challenging topics. The content knowledge and instructional strategies are well integrated in the tutorials in order to fulfill objectives mentioned above. The tutorials have been proven to be effective at addressing student conceptual and reasoning difficulties [40].

Superposition and Reflection of Pulses
Reflection and Transmission
Two-source Interference
Wave Properties of Light
Multiple-slit Interference
A Model for Single slit diffraction

Figure 1.1. Standard sequence of tutorials on waves and physical optics used in PHYS 123 prior to the development of curriculum in this study

Figure 1.1. shows the sequence of tutorials on waves and physical optics typically used in PHYS 123 prior to the development of curriculum in this study. The sequence typically starts with *Superposition and Reflection of Pulses* and *Reflection and Transmission,* which provide an opportunity for students to apply superposition to transverse pulses and build a model for reflection at boundaries. This is followed by a sequence of four tutorials on interference. This sequence starts with two-source interference in the context of water, in which students reason about superposition and interference using path length difference. *Wave properties of Light* is intended to help students make a transition to a more abstract context, light. *Multiple-slit interference* and *A model for single slit diffraction* are a continuation of the first two tutorials on interference. Path length difference and the resultant phase difference are key elements in the models of the entire sequence of tutorials. Phasors are not introduced in this sequence of tutorials, although many instructors and textbooks introduce phasors as an alternative approach to reason about interference.

PHYS 225:

Spin

Dirac Notation

Two-state Time Dependence

Superposition in Quantum Mechanics

PHYS 324:

Classical Probability

Treating Functions as Vectors

Representations of Wave functions

Time Dependence in Quantum Mechanics

Energy Measurements

Position, Momentum, and Energy Measurements

Angular momentum in quantum mechanics

Addition of angular momentum

Figure 1.2. Tutorials on quantum mechanics that have been frequently used in PHYS 225 and PHYS 324

Similar to an introductory-level tutorial section, PHYS 324 also has a 50-minute tutorial every week, while PHYS 225 uses the tutorials during lectures. Figure 1.2. shows the tutorials that are typically used in PHYS 225 and PHYS 324. The tutorials that are relevant to the study described in this dissertation are bolded.

There are three tutorials that target student understanding of vector spaces. In *Dirac notation,* students compare the inner products of Cartesian vectors and those of state vectors. *Treating functions as vectors* is intended to help students relate discrete and continuous cases with a focus on inner products of graphical functions. The primary goal of *Representations of wave functions* is to help students recognize that any representation of a quantum state provides information about probabilities for all observables.

Other bolded tutorials are related to superposition and time dependence. In *Superposition in quantum mechanics*, students consider how superposition states and the corresponding mixed states can be experimentally distinguished. The others are on time dependence and measurements. Although they focus on qualitative predictions about time dependence for probabilities, students are not guided explicitly to relate relative phases and probabilities. Many of these tutorials have been proven effective in addressing some aspects of student difficulty with superposition and with time dependence [11,79].

1.3 Research Methodology

In this section, we first discuss our iterative process of curriculum development, which is guided and validated by research on student understanding. We then describe the methods used for data analysis.

1.3.1 Iteration of Curriculum Development

The research involving curriculum development is an iterative process [56] with three components: (1) investigate student understanding, (2) develop curriculum, and (3) assess curriculum, as illustrated in Figure 1.3. These three components interact with and enhance each other discussed below.

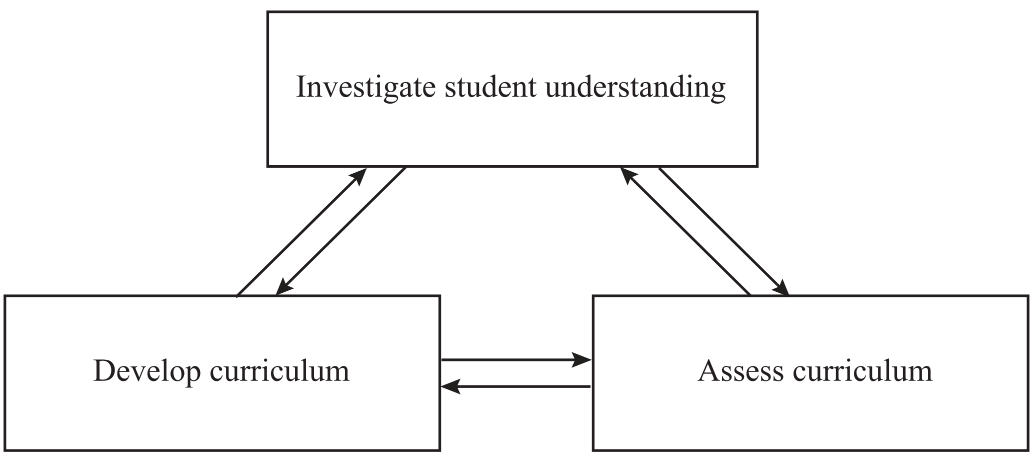


Figure 1.3. An interactive and iterative process of curriculum development.

We use qualitative (sometimes along with quantitative) tasks to probe student conceptual understanding and reasoning skills. Most of these tasks ask for explanations that allow us to gain insights into student thinking. Through analyzing student responses, we characterize and categorize common lines of reasoning in order to identify conceptual and reasoning difficulties. The patterns of student responses often guide us to modify the existing tasks (*e.g.,* vary the prompt or context) and/or to formulate new tasks that further probe student understanding from different perspectives. The research findings on student understanding guide curriculum development.

Through guided inquiry, our curricula are intended to help students develop coherent concepts, reasoning skills, and interpretation of physics formalism [56-57]. The sequences of scenarios and questions are carefully structured. The exercises often start with a relatively straight-forward scenario in which a basic concept or model can be developed. We gradually increase the complexity of scenarios that require student to apply the concept or model that they have built previously. We use a variety of specific instructional strategies, such as *observe-recognize-apply* [57], *elicit-confront-resolve* [58], and *apply-reflect-generalize* [56]. Through deliberately integrating the research findings and instructional strategies in curriculum, our understanding of student thinking is often reinforced.

The curriculum development is followed by assessment. The results of assessment guide us to modify or to revise the curriculum. Further investigations of student understanding are often needed to gain insights into what other possible intellectual or instructional approaches may be more effective.

1.3.2 Data Collection

Both qualitative and quantitative data are collected during the iterative process of curriculum development. The questions we use to probe student understanding in this dissertation are administered in three different settings: (1) online survey, (2) exam, and (3) interview.

An online survey, often called a tutorial pretest, is administered before each tutorial session. The tutorial pretests usually cover the material that has been already presented in lecture. The questions are usually written in the multiple-choice format followed by a long response for explanation. Students have limited time to complete the survey. The time limit varies depending on the course (usually 15 minutes for introductory-level courses and 25 minutes for upper-division courses). The pretests are graded for completion, but not for correctness. It has been our experience that most students, especially junior-level students, seem to take the pretests seriously. Some of the data presented in this dissertation suggest that student performance, at least for certain topics, on online survey and on exam has no significant difference if prior instruction is the same.

We also administer questions on exams. Sometimes we use exam questions to probe student understanding before curriculum is developed. Other times exam questions are used for assessment after students have received tutorial instruction and have completed tutorial homework. Analyzing student responses on exam questions informs us what difficulties have been effectively addressed, and what have persisted after lecture and/or tutorial instructions.

In the introductory-level courses, there have been two midterm exams and one final exam since 2017. Before 2017, there were three midterm exams. About twenty percent of the points on each exam are composed of tutorial-based questions. The tutorial-based questions on midterm exams are in long-response format. Students must explain their reasoning for full credit. On final exams, the questions are by default in multiple-choice format. The sophomore- and junior-level courses usually have two midterm exams and one final exam. All these questions are usually in long-response format. The proportion of points for tutorial-based questions vary depending on the class policy decided upon by the lecture instructor. The specific situations will be discussed in the relevant chapters.

We conduct individual student interviews to probe student understanding in depth. The interviewees involved in the study presented in this dissertation are student volunteers enrolled in PHYS 325. They are typically in the top half of their classes, but we do not select them or recruit them on this basis. The interviews are semi-structured and follow a think-aloud protocol. The semi-structured format allows us to investigate deeply a student's understanding when an interesting idea arises. The interview usually last from 40 to 60 minutes. The interviews are audio-taped for later transcription and analysis.

1.3.3 Data Analysis

We use both qualitative and quantitative methods to analyze data.

With interview transcripts, we use emergent coding [59] to identify students' lines of reasoning. In some cases, the interview data are used to confirm the inferences we make based on common student responses on written tasks (on exams or online surveys). In other cases, the interview data standalone. As will be discussed in detail in chapter 2, the analysis of the interview transcripts for this research is grounded in a theoretical framework developed by Gire and Price [33]. This framework categorizes four structural features of three common notations used in quantum mechanics. We use emergent coding to identify the instances in which the structural features can be regarded as supporting and/or hindering student reasoning.

With student written work, we characterize and categorize the apparent lines of reasoning. In this way, the prevalence of specific difficulties is documented. When certain patterns arise from the data (e.g., when the prevalence of correct answers on two analogous questions A and B appear similar or different), statistical tools are used to help us make inferences. In this dissertation, contingency tables are often used to display the frequency distribution of two categorical variables. For the example discussed above, the two categorical variables are questions (question A or B) and performance (correct or incorrect answer). We then use statistical tests to infer whether or not the prevalence of correct answers between questions A and B is different.

One of the common statistical tests used for categorical variables in this dissertation is Fisher's exact test [60]. We chose to use Fisher's exact test rather than a chi-square test [61] because

Fisher's exact test does not require an assumption that the distribution is normal and some of our data have small sample sizes so that the distributions are not necessarily normal. Fisher's exact test assumes that the individual data are independent and the total number of data in each category is fixed. The null hypothesis is that the proportions for one variable are the same for different values of the second variable (e.g., that the proportions of correct and incorrect answers for different questions, A and B, are the same). With Fisher's exact test, we can calculate the probability of getting the observed data, and all data sets with more extreme deviations, under the null hypothesis that the proportions are the same. We use the convention of a significance level of 0.05. That is, if the *p*-value is less than 0.05, we reject the null hypothesis; otherwise, we do not reject the null hypothesis and conclude that the difference between the proportions is not statistically significant (at the 0.05 level).

To assess the effectiveness of tutorial intervention, we often compare student performance in a class that had tutorial instruction to student performance in another class that did not have tutorial instruction. We would randomize students enrolled in different classes since this is beyond our control. However, prior research has shown that student performance from different classes after lecture but before tutorial instruction does not vary substantially [62]. We use Fisher's exact test since student performance in different classes can be considered as independent. Typically, we compare the prevalence of correct answer and the prevalence of correct answer with correct reasoning between classes. In this dissertation, we always use a two-tailed *p*-value instead of one-tail since students who received intervention can possibly perform worse than students who did not.

Another commonly used test in this dissertation is McNemar's test [63], which also does not require an assumption of data being normally distributed. We use McNemar's test when the two samples are not independent, but are two sets of paired data. For example, if we have the same class of students and we want to compare their performance before and after tutorial instruction, we use McNemar's test. An online survey students took before tutorial instruction (and usually after lecture instruction) serves as a tutorial pretest. A post-test could be exam questions or questions used in a different online survey after students received tutorial instruction on a relevant topic.

1.4 Organization of Dissertation

This dissertation is composed of three parts, each involves investigating and improving student understanding of a specific aspect of the superposition principle.

The first part extends from chapter 2 to chapter 3. Chapter 2 presents an investigation into student ability to determine quantum probabilities for both discrete and continuous cases. In chapter 3, we describe curriculum that has been developed based on the research findings in chapter 2. In addition, we present results from assessments of the curriculum.

The second part is on student ability to recognize the measureable effects of relative phases in superposition states. Chapter 4 presents the results from our study on this aspect. Chapter 5 discusses development and assessment of curriculum that is intended to address the issues discussed in chapter 4.

The third part is composed of chapter 6 and chapter 7. In chapter 6, we identify some of the difficulties introductory students encounter in using superposition to reason about interference of classical waves. Chapter 7 discusses the development and assessment of curriculum that we developed to help students reason about classical interference.

Lastly, we summarize our research findings and discuss future work in chapter 8.

Chapter 2. Student Understanding of the Use of Inner Products to Determine Quantum Probabilities

Superposition is a fundamental idea in quantum mechanics. The state of a particle is typically described by a state vector, which can be expressed as a superposition of basis states. This is similar to classical waves, which can be linearly combined. However, a quantum state does not predict the measurement results with certainty, but rather contains information about the probability of each result. The probabilities of measurement results depend on the coefficients for the component basis states. "Loosely speaking, c_n [the coefficient] tells you the 'amount of ψ_n [the component basis state] that is contained in ψ [the state][55].'" This coefficient can be determined using an inner product, as stated in the Born rule, discussed below.

The Born rule describes the method to determine probabilities for arbitrary observables with discrete and/or continuous eigenvalues. This rule is typically introduced at the beginning of textbooks on quantum mechanics, using either a position or a spin context. The statement of the rule is then followed by Born's generalized statistical interpretation typically given in a later chapter. McIntyre's textbook [54], for example, explicitly discusses several postulates that are related to the Born rule: (1) a physical observable is represented mathematically by an operator $\hat{A}$ that acts on kets, (2) the only possible result of a measurement of an observable is one of the eigenvalues a_n of the corresponding operator $\hat{A}$, and (3) the probability of obtaining the eigenvalue a_n in a measurement of the observable $\hat{A}$ (with discrete eigenvalues) on the system in the state $|\psi\rangle$ is $|\langle a_n|\psi\rangle|^2$. (Note that the inner product $\langle a_n|\psi\rangle$ results into the coefficient c_n.)

For observables with continuous eigenvalues, such as position, the coefficient $c(x)$ is a function of position. This coefficient can be expressed as an analogous inner product $\langle x|\psi\rangle$, where $\langle x|$ is the dual state of the position eigenstate. Since the wave function is defined as the probability amplitude associated with a position x, the coefficient $c(x)$ is the position-space wave function, which is typically written as $\psi(x)$. The probability of finding a particle in a small interval dx is $|\langle x|\psi\rangle|^2 dx = |\psi(x)|^2 dx$. Thus, the probability of finding the particle in the specified region from $x = x_1$ to $x = x_2$ is the integral, $\int_{x_1}^{x_2} |\psi(x)|^2 dx$. The Born's generalized statistical interpretation discussed above demonstrates that the formalism for determining probabilities in continuous and discrete cases are analogous. Taking an inner product is a key step for determining probabilities.

An understanding of Born's generalized statistical interpretation, is implicitly or explicitly emphasized by instructors. The University of Colorado – Boulder, for example, has developed learnings goals [64,65] for their upper-division quantum mechanics course. One of the learning goals is that students should be able to recognize quantum mechanics as a coherent and complete theory. The subject goals for formalism include students being able to calculate the probability for any common observable for any given quantum state. Thus, it is important that students are able to determine probabilities for discrete and continuous cases coherently.

Our study investigates student functional understanding of quantum states and inner products in vector spaces. In particular, we examine the extent to which students recognize that probabilities for observables with discrete and/or continuous eigenvalues can be determined using analogous procedures.

2.1 Prior Research

In chapter 1, we have discussed some overarching research findings on student understanding of quantum mechanics. In this section, we focus on prior research that we consider is closely related to student understanding of the use of inner products to determine probabilities. We start with discussing relevant prior research on introductory students' understanding of spatial vectors. We then review relevant prior research on student understanding of upper-division quantum mechanics.

Inner products with quantum state vectors are analogous to dot products with spatial vectors. Thus, it can be worthwhile to review research literature on student understanding of dot products with spatial vectors. Barniol and Zavala investigated student ability to calculate dot products and to interpret dot products as projections in physics and non-physics contexts [41]. They found that introductory students did relatively well on calculations of dot products, but less well on interpretation of dot products. They also found that students performed significantly better on interpretation in physics contexts than in non-physics contexts. In contrast, Emigh [66] found that the sophomore-level students in a modern physics course at the UW performed relatively well when they were asked to give an interpretation of a dot product in a non-physics context. Emigh, however, did not compare student performance in physics and non-physics contexts.

Although not much research explicitly focuses on inner products of quantum states, there have been a number of studies probing student understanding of measurements and probabilities [2-4,8,22]. Zhu and Singh [3,4] for example, found that when asked for position or energy probabilities, many students tend to believe that the expression for probability involves the corresponding operators. Those students seem to confuse the Hamiltonian acting on a state with an energy measurement. They also found that many students do not recognize that an energy probability can be determined using an inner product when the wave function is not explicitly written as a linear combination of energy eigenfunctions. Marshman and Singh [20,23,29] identified various difficulties that students have with Dirac notation. In particular, they found that student difficulty in determining probabilities arise not only when the expressions are written in terms of wave functions, but also when they are expressed in Dirac notation. Many students make errors on tasks that requires translation between different notations.

Prior research [33,67] has demonstrated in both introductory and upper-level courses that student performance can be influenced by different representations and/or notations. Heckler and Scaife, for example, found that student performance on tasks involving adding and subtracting vectors in (algebraic) *ijk* format is significantly better than that in (graphical) arrow format [67]. Gire and Price [33] have discussed the structural features of different common algebraic quantum notations. For example, Dirac notation is compact and provides high symbolic support for computation, whereas wave function notation provides little support for computation. However, wave function notation contains detailed information about the mathematical behavior of the system. Gire and Price found that when asked to write an expression for expectation value, interviewees tend to use Dirac notation as a template and make fewer errors using Dirac notation.

2.2 Overview and Goals

This section provides an overview of the study presented in this chapter. The primary goals of this study are to examine (1) student functional understanding of quantum states and inner products in vector spaces and (2) the extent to which students recognize that probabilities for observables with discrete and/or continuous eigenvalues can be determined using analogous procedures. We use both quantitative and qualitative methods to probe student functional understanding of quantum states and inner products. The quantitative study is discussed in sections 2.3, 2.4 and 2.5, and the qualitative study is described in section 2.6.

As discussed in section 2.1, prior research has documented student difficulty with determining probabilities in quantum mechanics. However, we are not aware of any studies that specifically focus on the extent to which students recognize that probabilities for observables with discrete and/or continuous eigenvalues can be determined using analogous procedures. Taking an inner product is a key step in both discrete and continuous cases. In order to examine the extent to which students recognize that taking an inner product is necessary for determining probabilities, we conducted a quantitative study involving three major aspects: (1) student proficiency in performing computations, (2) student ability to apply inner products in different contexts (or on different types of tasks), and (3) student ability to recognize that any representation (e.g., the energy or position representation) of a quantum state provides information about probabilities for all observables. These three aspects require different levels of understanding of quantum states and inner products.

Section 2.3 presents tasks that primarily require proficiency in performing computations, which illustrates the lowest level of understanding. We assess the extent to which students are able to compute inner products, find orthogonal states, and determine probabilities for observables with discrete eigenvalues, such as spin. All the tasks were administered in the sophomore-level course.

In section 2.4, we discuss student performance on tasks that require an intermediate level of understanding. We first discuss (sophomore-level) student performance on tasks that require inferences about quantum states from statistical measurement results in section 2.4.1. We then, in section 2.4.2, analyze junior-level students' ability to determine probabilities for angular momentum.

The highest level requires students to have a functional understanding of quantum states and inner products in vector space, which is essential for determining probabilities for discrete and continuous cases using analogous procedures. We start with analyzing (sophomore- and junior-level) student performance on tasks that involve translating expressions of quantum states and inner products between wave function and Dirac notation in section 2.5.1. We then discuss the performance of junior-level students on tasks that require changing between energy and position representations in sections 2.5.2 and 2.5.3. The primary goal is to examine to what extent junior-level students recognize that determining probabilities involves taking inner products. The results motivated a further investigation (discussed in section 2.5.4) into the extent to which junior-level students recognize that any representation of a quantum state provides information about probabilities for all possible observables.

Some of the findings from our quantitative study suggest that the features of different quantum notations may have an impact on student performance, which is consistent with the results of Gire and Price [33]. Therefore, we followed up by conducting interviews with junior-level students. The *structural features of quantum notations* framework guided our analysis of interview transcripts. Section 2.6.1 provides a review of the theoretical framework, *structural features of* *quantum notations*. In section 2.6.2, we discuss example excerpts from interview transcripts that illustrate how structural features of different notations can be regarded as supporting and/or hindering student sense-making of quantum states and inner products.

2.3 Student Proficiency with Basic Computations

To investigate student proficiency with basic computations, we have given students various tasks that prompt them to calculate inner products and probabilities. Most of these tasks were given in a spins context, although a few were based on hypothetical scenarios involving fictitious observables that have discrete eigenvalues. All tasks but one were quantitative (i.e., numerical answers were required). The primary objective is to assess the extent to which students are able to perform the important steps required for computing these quantities. We define important steps as: (1) changing basis (for spin states), (2) taking complex conjugates when performing inner products and (3) taking complex conjugates when performing modulus squares. We consider these important because they are necessary for applying inner products in the qualitative tasks discussed in the subsequent sections of this chapter.

All tasks in this section were administered on exams in PHYS 225. The lecture had covered the first two chapters in McIntyre textbook: Stern-Gerlach experiments, operators, and measurements. Students had also completed a preliminary version of tutorial *Dirac notation*. The primary goals of this tutorial were to help students understand quantum state vectors and to become fluent with Dirac notation. It is worthwhile to note that an equation sheet was provided by the instructor so that students were not required to memorize the definitions of relevant concepts. In Winter 2017 and Winter 2018, students were asked to prepare and to bring their own equation sheets.

2.3.1 Computations of Inner Products

Figure 2.1 shows two tasks that require computing inner products and magnitudes of states vectors. In Task 2.1, students consider two spin-1/2 systems. One state is written in the S_z basis and the other is written in S_x basis. Students are asked to compute the inner product between these two spin states. Task 2.2 involves two state vectors in a 4-dimensional Hilbert space. In part A, students are asked to compare the magnitudes of those two states vectors. In part B, students are asked to compute an inner product.

Task 2.1

Spin ½ systems are prepared in these states:
$|\alpha\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$ and $|\gamma\rangle = \frac{1}{\sqrt{2}}|+\rangle_x - \frac{i}{\sqrt{2}}|-\rangle_x$.
Note that the states are not expressed in the same basis.
Compute $\langle\alpha|\gamma\rangle$. Show your work and/or explain your reasoning.

Task 2.2

Consider a 4-dimenisonal Hilbert space. The basis vectors $|\chi_i\rangle$ are orthonormal. The states vectors are defined below:

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle \qquad |B\rangle = 2i|\chi_1\rangle + |\chi_2\rangle + |\chi_4\rangle$$

A. Is the magnitude of state vector $|A\rangle$ greater than, less than, or equal to the magnitude of state vector $|B\rangle$? Explain.

B. Evaluate the expression $\langle B|A\rangle$. Show your work.

Figure 2.1. Tasks about inner products on midterm exams given in PHYS 225.

In Task 2.1, states $|\alpha\rangle$ and $|\gamma\rangle$ are expressed in different bases ($|\alpha\rangle$ in the S_z basis and $|\gamma\rangle$ in the S_x basis), thus one needs to change basis (explicitly or implicitly) for either state. It might be easier to change from S_z basis to S_x basis, since $|\alpha\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle = |-\rangle_x$. Thus, $\langle\alpha|\gamma\rangle = \frac{1}{\sqrt{2}}\,{}_x\langle -|+\rangle_x - \frac{i}{\sqrt{2}}\,{}_x\langle -|-\rangle_x = -\frac{i}{\sqrt{2}}$. In Task 2.2, the states are expressed in the same basis. To determine the magnitude of either state vector, one can take the square root of the inner product between the state vector and itself (*i.e.*, $\sqrt{\langle A|A\rangle}$). Taking complex conjugates is required when computing the inner products since the magnitudes of state vectors can only be real and positive numbers (or zero). The magnitudes of both $|A\rangle$ and $|B\rangle$ are $\sqrt{6}$, so they are equal.

Table 2.1. Student performance on Tasks 2.1 and 2.2

	Task 2.1	Task 2.2A	Task 2.2B
Correct answer (N = 130[a])	58%	52%	67%

[a]PHYS 225: WIN15.

Student performance is shown in Table 2.1. On Task 2.1, only 58% of students gave a correct answer. About 22% of students correctly wrote down the relation between S_z and S_x basis states, but made errors for the multiplications. About 8% of the students treated the spin states as if they were expressed in the same basis. It is likely that they overlooked subscripts, or that they did not recognize that there can be different sets of basis states. About 12% incorrectly related these different basis states (*e.g.,* incorrectly wrote that $|+\rangle_x = \frac{1}{\sqrt{2}}|\uparrow\rangle$) or made other errors. Recall that an equation sheet was provided by the instructor so that students were not required to memorize the relations between different basis. The results seem to suggest that some students have difficulty with changing basis even after lecture and tutorial instruction.

On Task 2.2, part A, most of the students explicitly computed the magnitudes of both state vectors and then compared their magnitudes. Some students explicitly wrote down expressions involving inner products (e.g., $\sqrt{\langle A|A\rangle}$); others directly computed the sum of the modulus square for each coefficient. About half of the students gave a correct answer. Many students (26%) computed the *square* instead of the modulus square for each coefficient. About 3% of the students wrote down the inner products, but did not take the complex conjugate for the coefficients. It seems that these students were treating the quantum state vectors like spatial vectors, the coefficients of which are real.

The performance on part B was slightly better than that on part A. About 67% of the students gave a correct answer. The common errors for part B were similar to that for part A. About 9% of the students did not take the complex conjugate for $\langle B|$. Other common errors involve algebra mistakes. Comparing student performance and common errors on both parts, it seems that students were more likely to treat quantum state vectors like spatial vectors (i.e., neglect to take the complex conjugate, or compute the square instead of the modulus square) when they were not prompted to compute an inner product. This may be due to the fact that the structure of the inner product in Dirac notation provides a visual cue for students to take the complex conjugate of the coefficients for the dual state.

2.3.2 Computations of Orthogonal States

Figure 2.2 shows two tasks that require students to find states that are orthogonal to given states. Task 2.3 has two versions. Both versions of Task 2.3 ask students to write the dual state for the given spin state, and then find a state that is orthogonal to the given spin state. The only difference between the two versions are the forms of the complex coefficients: on version α, one of the coefficients is expressed as a complex exponential with a generic argument $-\varphi$; on version β, the coefficients have the rectangular form (*i.e.,* $x + iy$). To determine the dual state, one needs to take the complex conjugate of the coefficients, and then change the kets into bras. For example, the dual state for $|\psi\rangle = |+\rangle - e^{-i\varphi}|-\rangle$ is $\langle\psi| = \langle+| - e^{i\varphi}\langle-|$. To find a state $|\phi\rangle$ that is orthogonal to $|\psi\rangle$, one can let $\langle\psi|\phi\rangle = 0$ and then find the relation between the coefficients of state $|\phi\rangle$. Since the question does not require state $|\phi\rangle$ to be normalized, one may choose to assign a random value to one of the coefficients, and then determine the other coefficient. Thus, there are many different answers for this question.

Compared to Task 2.3, Task 2.4 is less open-ended. Students were given two state vectors, and one of them has an unknown coefficient. It asks for the unknown coefficient such that the two state vectors are orthogonal. One can use $\langle A|C\rangle = 0$ to uniquely determine the unknown coefficient.

Task 2.3

Version $\boldsymbol{\alpha}$: A spin state is described in the S_z basis by the ket $|\psi\rangle = |+\rangle - e^{-i\varphi}|-\rangle$.

Version $\boldsymbol{\beta}$: A spin state is described in the S_z basis by the ket $|\psi\rangle = -\sqrt{\frac{3}{5}}|+\rangle + i\sqrt{\frac{2}{5}}|-\rangle$.

A. Write the corresponding dual state (bra).

B. Write a state that is orthogonal to the original state $|\psi\rangle$.

Task 2.4

Consider a 4-dimenisonal Hilbert space. The basis vectors $|\chi_i\rangle$ are orthonormal. The states vectors are defined below:

$$|A\rangle = |\chi_1\rangle + 2|\chi_3\rangle + i|\chi_4\rangle \qquad |C\rangle = i|\chi_2\rangle - i|\chi_3\rangle + \gamma|\chi_4\rangle$$

Determine the value γ of such that state vector $|C\rangle$ is orthogonal to state vector $|A\rangle$.

Figure 2.2. Tasks about orthogonal states on midterm exams in PHYS 225.

Student performance on different versions of Task 2.3 is similar, as shown in Table 2.2. About two-thirds (67% on version α and 76% on version β) of the students wrote the correct dual state, and only about one-third (33% on version α and 30% on version β) wrote a correct orthogonal state. Similar to student responses to Task 2.1 and 2.2, many students made errors suggesting that they lack fluency with basic algebra. However, we found that there was a common error related to the complex coefficients. Some students failed to write the correct complex conjugates for the dual states. Moreover, many students (16% on version α and 23% on version β) wrote a state that would be orthogonal to the given state if taking complex conjugate is not required. Similarly, about 12% of the students neglected to take the complex conjugate on Task 2.4.

Table 2.2. Student performance on Tasks 2.3 and 2.4

	Task 2.3, version α $N = 130$[a]		Task 2.3, version β $N = 126$[b]		Task 2.4 $N = 130$[a]
	Dual state	Orthogonal	Dual state	Orthogonal	Orthogonal
Correct answer	67%	33%	76%	30%	45%
A Common error: no complex conjugate	13%	16%	5%	23%	12%

[a]PHYS 225: WIN15. [b]PHYS 225: WIN16.

About one-third of the students showed their work in both versions of Task 2.3. We found that many students did not recognize that part A was intended to help them answer part B. For students who showed their work, about 20% (of the entire population) did not use their answer for the dual state. These students wrote expressions such as $\langle\phi|\psi\rangle = 0$. In this case, they would need to determine the coefficients for the ket $\langle\phi|$ and then take the complex conjugate to get the coefficients for the bra $|\phi\rangle$. This approach requires more work than using $\langle\psi|\phi\rangle = 0$ since they have already written $\langle\psi|$ in part A. Only a few students (about 5% of the entire population on both versions) found a correct orthogonal state when they used $\langle\phi|\psi\rangle = 0$.

We have found that many students consistently neglected to take the complex conjugate of complex coefficients in various tasks. These students seem to treat quantum state vectors as spatial vectors. Figure 2.3 shows an example student response on Task 2.3, version α. This student first rewrote the state vectors as column vectors. They explicitly wrote *dot* product instead of inner product of state vectors, and then simply multiplied the coefficients of the state vectors without taking complex conjugate.

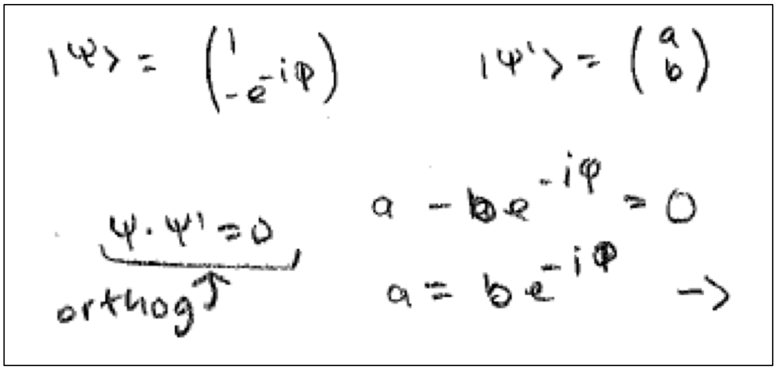


Figure 2.3. An example student response on Task 2.3, version α

It is possible that some students simply forgot to take complex conjugates. However, when students were prompted to compute an inner product, less students neglected to take the complex conjugate, as discussed previously on Task 2.2. It is likely that some prompts, such as magnitude (as in Task 2.2) and orthogonality, reinforce this type of error. With spatial vectors, orthogonal vectors are vectors that are perpendicular to each other, and magnitude means the length of a vector. The geometric interpretations of these terms may support students with the algorithm for spatial vectors. When students encounter abstract quantum state vectors, it is likely that they think about the geometry interpretations. However, the difference between treating quantum state vectors and spatial vectors may seem subtle to students. The evidence seems to suggest that many students have not developed the expertise to treat quantum state vectors properly.

2.3.3 Computations of Probabilities

Task 2.5 requires determining probabilities for spin-1/2 systems, as shown in Figure 2.4. Students consider three spin states. They are asked to rank the probability of obtaining $+\frac{\hbar}{2}$ in the z-direction. One of them is expressed in the S_z basis and the others are in the S_x basis.

Task 2.5

Spin ½ systems are prepared in these states:

$|\alpha\rangle = \frac{1}{\sqrt{2}}|\uparrow\rangle - \frac{1}{\sqrt{2}}|\downarrow\rangle$, $|\beta\rangle = \sqrt{\frac{3}{5}}|+\rangle_x - \sqrt{\frac{2}{5}}|-\rangle_x$, and $|\gamma\rangle = \frac{1}{\sqrt{2}}|+\rangle_x - i\frac{1}{\sqrt{2}}|-\rangle_x$.

If the spin was measured along the *z*-direction, rank the probability of obtaining $+\frac{\hbar}{2}$ for each of the three prepared states. If any of the states have a probability equal to zero or one, state one explicitly. Show your work and/or explain your reasoning.

Figure 2.4. A task about spin probabilities on a midterm exam in PHYS 225.

The spin probabilities can be determined by the modulus square of inner products. For example, the probability of measuring $+\frac{\hbar}{2}$ in the *z*-direction for state $|\gamma\rangle$ is $P\left(S_z = \frac{\hbar}{2}\right) = |\langle\uparrow|\gamma\rangle|^2$. Since $|\gamma\rangle$ is written in the S_x basis, one needs to change basis. Plugging in $\langle\uparrow| = \frac{1}{\sqrt{2}}\,{}_x\langle+| + \frac{1}{\sqrt{2}}\,{}_x\langle-|$, we will get $P\left(S_z = \frac{\hbar}{2}\right) = \left|\frac{1}{2} - i\frac{1}{2}\right|^2 = \frac{1}{2}$.

Most of the students computed the probability for each state in Task 2.5 before they provided the rank of the probability. In order to make a direct comparison of student proficiency in computing each probability, we present the percentage of correct probability for each state in Table 2.3.

Table 2.3. The percentage of each correct probability in Task 2.5

	$\|\alpha\rangle$	$\|\beta\rangle$	$\|\gamma\rangle$
Correct probability (N = 130[a])	81%	41%	45%

[a]PHYS 225: WIN15.

About four-fifths of the students correctly computed the probability for state $|\alpha\rangle$, which does not require changing basis. However, only about two-fifths of the students computed a correct probability for either state $|\beta\rangle$ or $|\gamma\rangle$. Both of these probabilities require changing basis. The data show that students performed better computing a probability that does not require changing basis than a probability that does.

Many (17%) students made algebraic errors when they computed $\left|\sqrt{3}-\sqrt{2}\right|^2$ for state $|\beta\rangle$. It is common (12% of all students) that students wrote $(\sqrt{3}-\sqrt{2})^2 = 1$. It seems that these students incorrectly believed that $(\sqrt{3}-\sqrt{2})^2 = (\sqrt{3})^2 - (\sqrt{2})^2$.

When determining the probability for state $|\gamma\rangle$, many (14%) students made an error on the modulus square $\left|\frac{1}{2} - i\,\frac{1}{2}\right|^2$. Some (5% of all students) determined that the modulus square is zero. It is likely that these students made the same error as those who wrote $(\sqrt{3}-\sqrt{2})^2 = 1$. We also found that some (4% of all students) treated the modulus square as a "regular" square, which is consistent with what we found in sections 2.3.1 and 2.3.2.

Another common (10% of all students) error is neglecting to change basis or incorrectly changing basis. We have also discussed this error in sections 2.3.1 and 2.3.2.

We have identified several common errors students made when they were computing inner products and probabilities. The common errors involve neglecting to change basis, failing to take complex conjugates for inner products and for modulus squares, and other algebraic errors.

2.4 Student Ability to Apply Inner Products in Different Contexts

Not only are students expected to be fluent with basic computations involving inner products, but they also are required to apply the same concept in a variety of physics scenarios. In this section, we assess student ability to apply inner products in different contexts.

In section 2.4.1, we analyze sophomore-level students' responses to questions that require inferring quantum states from statistical measurement results. These tasks are slightly more challenging since they require students to be able to write possible quantum states based on statistical results, rather than just plug-and-chug. In section 2.4.2, we assess junior-level students' ability to determine probabilities in a more advanced context: angular momentum. The lecture and textbook tend to focus on tasks that require calculus with orbital wave functions for angular momentum. Determining probabilities for different components of angular momentum is not emphasized in lecture. Thus, we think this context allows us to probe to what extent students are able to apply inner products.

2.4.1 Infer Quantum States from Statistical Measurement Results

We designed a task to examine student ability to infer quantum states from statistical measurement results in a hypothetical context. The task was administered on final exams in PHYS 225. As shown in Figure 2.5, Task 2.6 describes a hypothetical scenario of a quantum mouse. It assumed that there were two physical quantities that we can observe on a quantum mouse: mood and weight. Each observable has two eigenvalues. The eigenequations for these observables are provided. Note that this system (the quantum mouse) is analogous to a spin-1/2 system. Students are given a quantum state that describes many identical copies of the mouse. The weight of each mouse was measured to be 1 (i.e., "light"). It asks for the inner product between a "light" mouse and a "happy" mouse. Versions with different coefficients of the quantum state were given in different academic quarters.

Task 2.6
Consider a quantum mechanical mouse. You can observe the weight of the mouse and the mood of the mouse. There are only two possibilities for the weight, the two possibilities for the mood. The eigenequations for the weight are: $\widehat{W}\|light\rangle = 1\|light\rangle$, and $\widehat{W}\|heavy\rangle = 5\|heavy\rangle$. The eigenequations for mood are: $\widehat{M}\|happy\rangle = 1\|happy\rangle$ and $\widehat{M}\|sad\rangle = -1\|sad\rangle$. Assume that each set of eigenstates is orthonormal.
Consider many identical copies of the mouse described by the state $\|\phi\rangle = \frac{3}{5}\|happy\rangle - \frac{4i}{5}\|sad\rangle$. Suppose you measured the weight of each mouse and determined that every mouse had a weight of 1 (i.e., "light").
Determine the value of the inner product between a "light" mouse and a "happy" mouse? Explain how you arrived at your answer.

Figure 2.5. A task involving inferring quantum states from measurement results given on final exams in PHYS 225.

To answer this question, students need to infer the state vector from the statistical measurement results. Since each mouse has a weight of 1, the state $|\phi\rangle$ is equal to state $|light\rangle$ (or $|\phi\rangle = e^{i\theta}|light\rangle$, where θ is an arbitrary angle). Thus, $\langle light|happy\rangle = \langle\phi|happy\rangle = \frac{3}{5}$.

Since student performance did not vary significantly over different academic quarters, we aggregated the results, as shown in Table 2.4. About half (51%) of the students gave a correct answer for the inner product. About two-fifths of the students (of the entire population) correctly explained how they determined their answer. About 14% of the students, although they recognized that the inner product $\langle\phi|light\rangle = 1$, were not able to determine the inner product $\langle light|happy\rangle$.

About 10% of the students determined that the inner product $\langle light|happy\rangle = 1$. Some of them explained that since states $|light\rangle$ and $|happy\rangle$ share the eigenvalue, the inner product is one. These students seemed to confuse eigenvalues with coefficients of quantum states. Some of the students expressed $|light\rangle$ and $|happy\rangle$ in the same basis (they used the same column vector $\binom{1}{0}$ to represent both states), and determined the inner product to be one.

About 8% gave an answer that was equal to the probability (of a light mouse to be happy) instead of the inner product. Some of them answered $\frac{3}{5}\langle happy|light\rangle$; and some answered that the probability of finding a light mouse to be happy is $\frac{9}{25}$, thus the inner product is $\frac{9}{25}$. It seemed that these students confused probabilities with inner products.

Table 2.4. Student performance on Task 2.6

	Correct answer	Correct answer with reasoning	Recognize $\langle\phi|light\rangle = 1$ but incorrect answer
$N = 273$[a]	51%	40%	14%

[a]PHYS 225: WIN15 and WIN17.

2.4.2 Apply Inner Products in Angular Momentum Context

We administered a task to assess student ability to use inner products to determine angular momentum probabilities, as shown in Figure 2.6. It was administered on a final exam in PHYS 324 in Autumn 2014. The lecture and textbook focused on tasks that require complicated mathematical calculations. However, students had seen similar tasks in tutorials that are intended to help students build a model for angular momentum.

Task 2.7

Recall that $\vec{J} = \vec{L} + \vec{S}$ represents the total angular momentum of a system. Suppose a particle is in the state $|l, s; j, m_j\rangle = |3,1; 4,1\rangle$.

Suppose that the x-component of the total angular momentum (J_x) is measured to be $+\hbar$. After this measurement, determine the possible results of a measurement of the z-component of the total angular momentum (J_z), and describe how to calculate the probability of each result. Explain your reasoning.

Figure 2.6. A task about measurement outcomes of angular momentum and probabilities on a final exam in PHYS 324.

In Task 2.7, a quantum state that describes the particle was written in the J_z basis. The J_x of the particle was measured to be $+\hbar$. Students were asked to determine the possible outcomes for J_z after the measurement of J_x and their associated probabilities. After the measurement of J_x, the particle is in an eigenstate of J_x. Since J_x and J_z do not commute, J_z does not have a well-defined eigenvalue. In principle, the possible outcomes are any of the allowed eigenvalues of J_z, which goes in integer steps from $-j$ to j times $\hbar$. To determine the probabilities, one can use the inner products. For example, the probability of measuring $-4\hbar$ is given by $\left| {}_z\langle 4,-4|4,1\rangle_x \right|^2$.

Table 2.5. Percentage of correct answer for each question in Task 2.7

	Possible outcomes	Probabilities
Correct answer ($N = 84$[a])	32%	25%*

[a]PHYS 324: AUT14. *about 10% (of the entire population) neglected to include the modulus square.

The percentages of correct possible outcomes and probabilities are shown in Table 2.5. About one-third of the students correctly determined the possible outcomes, and about one-fourth correctly described how to determine the probabilities. About 10% (of the entire population) correctly answered that one needs to take the inner products (or to write the J_x eigenstate in terms of the J_z eigenstates and to find the coefficients), but they did not mention that the modulus squares of inner products or coefficients were necessary. Since these students correctly described the key step, inner product, their answers were considered as correct.

Table 2.6. Percentages for incorrect answers for description of determining probabilities

	Use Clebsch-Gordan table or ladder operators	Others	Blank
Incorrect answers (N = 84[a])	15%	30%	30%

[a]PHYS 324: AUT14.

Table 2.6 shows the incorrect answers students made. About 15% of students answered that one can find the coefficients using a Clebsch-Gordan table or ladder operators on the J_x eigenstate. The Clebsch-Gordan coefficients (or ladder operators) can be used to relate the $|l, s; j, m_j\rangle$ basis to the $|l, m_l; s, m_s\rangle$ basis. However, they do not apply for the coefficients that relate the J_x and J_z bases. About 30% of the students did not describe how to determine the probabilities, but only about 6% of the students did not answer either part of this task. It seemed that many students did not know how to determine the probabilities. The results suggest that many students had difficulty applying inner products to determine the angular momentum probabilities.

2.5 Student Understanding of Vector Space and Representations of Quantum States

The concept of vector space is fundamental to quantum theory. The state of a particle is typically represented by a state vector in a vector space. The eigenstates of an observable form a complete basis. Thus, a state vector $|\psi\rangle$ can be written, in general, as a superposition (linear combination) of the eigenstates of an observable. The probability for a measurement outcome of an observable depends on the corresponding expansion coefficient. The expansion coefficient can be determined by taking the inner product between the eigenstate and the state vector.

Expressions for state vectors written in different bases are equivalent. For example, the state vector of a particle in a potential well can be written in terms of the energy eigenstates: $|\psi\rangle = \sum_n c_n|\varphi_n\rangle$. It can be also written in terms of the position eigenstates: $|\psi\rangle = \int_{-\infty}^{+\infty} \psi(x)|x\rangle dx$. However, a position-space wave function is more commonly used for a particle in a position-dependent potential. The position-space wave function $\psi(x)$, can be expressed as the inner product between the position eigenstate and the state vector, $\langle x|\psi\rangle$. Since the state vector can be written equivalently in any other basis, such as the energy basis, the wave function can be written in terms of the energy eigenfunctions. Thus, the information of probabilities for other observables (e.g., energy) can also be obtained from the position-space wave function. A quantum state described in a particular representation (e.g., energy/position representation) contains the information of probabilities for other observables as well.

In this section, we are investigating student functional understanding of vector spaces. We start with probing student ability to translate inner products between wave function and Dirac notation in section 2.5.1. We then examine student ability to determine energy probabilities given (algebraic or graphical) position-space wave functions that are not expressed as linear combinations of energy eigenfunctions in sections 2.5.2 and 2.5.3. Finally, in section 2.5.4 we analyze the extent to which students recognize that any representation of a quantum state provides information about probabilities for all (possible) observables.

[Part of this section is intended to be submitted to Physical Review Physics Education Research.]

2.5.1 Translating Inner Products Between Wave Function and Dirac Notation

Quantum mechanics involves a variety of representations, such as functions, graphs, matrices, and Dirac notation. A functional understanding of quantum states and inner products requires being able to translate between representations and notations. We designed tasks to probe student ability to translate between wave function and Dirac notation, two common notations used for contexts involving position-dependent potentials.

2.5.1.a Tasks design and administration

Figure 2.7 shows three tasks that require students to translate between wave function and Dirac notation. Task 2.10 has two versions. Version A is in long-answer format and Version B is in multiple-choice format. The answer options for the multiple-choice version were based on the common incorrect answers given on the long-answer version.

Version A of Task 2.9 was given on a final exam in PHYS 225 and Version B of Task 2.9 was given in online surveys in PHYS 225. Task 2.10 and Task 2.11, along with Version B of Task 2.9 were given in PHYS 324 in online surveys after students received lecture instruction on the infinite square well potential and before Dirac notation in PHYS 324. It is worthwhile to mention that students had been taught Dirac notation in PHYS 225.

Task 2.9

The state of a particle in an infinite square well at $t = 0$ can be described as $|\psi\rangle = \frac{1}{\sqrt{2}}|\varphi_1\rangle + \frac{1}{\sqrt{2}}|\varphi_2\rangle$, where $|\varphi_n\rangle$ represents the n^{th} energy eigenstate.

Version A:
Write an expression for the probability density in position space associated with this particle at $t = 0$.

Version B:
Which of the expression(s) below can represent the probability density in position space of the particle at $t = 0$? Check all that apply. (Assume the wave functions below are all real.)

A. $\langle\psi|\psi\rangle$

B. $\psi^2(x)$

C. $\frac{1}{2}\varphi_1^2(x) + \frac{1}{2}\varphi_2^2(x)$

D. $\frac{1}{2}\varphi_1^2(x) + \frac{1}{2}\varphi_2^2(x) + \varphi_1(x)\varphi_2(x)$

Task 2.10

Indicate all of the following equations that are correct.
(a) $\langle x|\psi\rangle = \int x\psi(x)dx$
(b) $\langle x|\psi\rangle = \int \delta(x - x')\psi(x')dx'$
(c) $\langle\varphi_n|\psi\rangle = \int \varphi_n^*(x)\psi(x)dx$

Task 2.11

Indicate all of the following equations that are correct.
(a) $|\psi\rangle = \int |p\rangle\langle p|\psi\rangle dp$
(b) $|\psi\rangle = \int \psi(x)|x\rangle dx$

Figure 2.7. Tasks requiring translating between Dirac and wave function notations

In Task 2.9, students are given a quantum state written in terms of energy eigenstates for an infinite square well. They are asked to write an expression for the probability density in position space. The probability density is defined as the modulus square of the wave function. Since the quantum state is written in Dirac notation, one needs to convert the quantum state from Dirac notation to wave function notation: $\psi(x) = \frac{1}{\sqrt{2}}\langle x|\varphi_1\rangle + \frac{1}{\sqrt{2}}\langle x|\varphi_2\rangle = \frac{1}{\sqrt{2}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$. By convention, the energy eigenfunctions for a particle in the infinite square well are usually taken to be real functions. Thus, the probability density is $|\psi(x)|^2 = \frac{1}{2}\varphi_1^2(x) + \frac{1}{2}\varphi_2^2(x) + \varphi_1(x)\varphi_2(x)$. In the variants of Task 2.9, students were asked to choose expression(s) either for probability density or for the probability associated with a specified region.

Task 2.10 asks students to indicate the equation(s) that correctly translate the inner product from Dirac to wave function notation. Although this task does not directly ask for probability, it probes whether students recognize that the wave function is an inner product and whether students are able to translate the inner product from Dirac to wave function notation. To answer this question, students need to identify the corresponding position-space wave functions for states $|x\rangle$ and $|\psi\rangle$. For a position eigenstate $|x'\rangle$, the position-space wave function is a one-dimensional delta function since $\langle x|x'\rangle = \delta(x - x')$. The result of the inner product between two state vectors is equal to the inner product between the corresponding two wave functions. Alternatively, students can use the definition of the wave function in Dirac notation, $\psi(x) = \langle x|\psi\rangle$ and check whether the integral results in $\psi(x)$. Answer (b) is therefore correct. Answer (c) is also correct since the position-space energy eigenfunction is $\varphi_n(x)$. In one of the academic quarters, answer (c) was not included in the task.

In Task 2.11, expressions of a quantum state vector written in position and momentum bases are given. Students need to understand that the position and momentum bases each form a complete basis and thus the state vector can be expressed as a superposition of position or momentum eigenstates. Both position and momentum have continuous eigenvalues, so the superposition should be expressed as integrals. The coefficients can be written as the inner products between the position/momentum eigenstate with the state vector or the wave function in position or momentum space. Both of the expressions are correct.

2.5.1.b Overall performance

Table 2.7 shows the percentage of correct answer for each task. The data from different versions of Task 2.9, and Task 2.10 are aggregated, respectively, since the results do not seem to vary from the long-answer format to the multiple-choice format.

On Task 2.9, about one-third of the students gave a correct answer. We defined correct answer as responses that involve at least one correct expression but no incorrect expressions. It is very common that students wrote or chose the expression $\langle\psi|\psi\rangle$.

Table 2.7. Student performance on Tasks 2.9, 2.10, and 2.11

	Task 2.9 (N = 382[a])	Task 2.10 (N = 98[b])	Task 2.11 (N = 77[c])
Correct answer	31%	26%	8%

[a]PHYS 225: WIN16, SUM16, WIN17, and SUM17. PHYS 324: SUM16, AUT16, SUM17.
[b]PHYS 324: AUT16 and SUM17. [a]PHYS 324: AUT16

On Task 2.10, less than one-third of the students recognized the expression that correctly translates inner product $\langle x|\psi\rangle$ from Dirac notation to an integral. About half of the students included the incorrect answer $\langle x|\psi\rangle = \int x\psi(x)dx$ in their responses. Some students regarded that both of the answers are correct.

Only about 8% of the students recognized that both expressions in Task 2.11 are correct. About two-fifths of the students only chose one of the expressions. Another two-fifths did not choose either of them.

2.5.1.c Identification of student difficulties

We discuss specific difficulties students have with translating expressions between wave function and Dirac notations below.

2.5.1.c.1 Tendency to confuse quantum state vectors with wave functions

On all versions of Task 2.9, most students correctly identified that the modulus square of the wave function, $|\psi(x)|^2$, is the probability density. However, many students also stated (incorrectly) that the probability density can be expressed as the inner product of the state vector with itself, $\langle\psi|\psi\rangle$. These students appear to believe that $\langle\psi|\psi\rangle$ and $|\psi(x)|^2$ are equivalent expressions written in different notations. When sophomore students were asked to write an expression for the probability density on the final exam, about one-third (30%) of the students (N=118) wrote incorrect expressions such as $|\psi(x)|^2 = \langle\psi|\psi\rangle$ and $|\psi(x)|^2 = \big||\psi\rangle\big|^2$. About 12% of the students concluded that the probability density is equal to one (or some other constant) after performing calculation on $\langle\psi|\psi\rangle$.

When Task 2.9 was given as a multiple-choice question where students were allowed to choose more than one answer, about half of the students included $\langle\psi|\psi\rangle$ (often along with a correct answer choice). A student response is quoted below: "It [$\langle\psi|\psi\rangle$] is [the probability density] in the Dirac notation, where the bra-ket represents the product of the original wave function and its complex conjugate." This student appears to be making a comparison between the two expressions based on their structures. The inner product $\langle\psi|\psi\rangle$ can be considered as the product of the bra $\langle\psi|$ (the dual state of $|\psi\rangle$) and the ket $|\psi\rangle$. Similarly, the modulus square of the wave function is the product of $\psi^*(x)$ (the complex conjugate of wave function) and $\psi(x)$. Both expressions are products of a mathematical object and its conjugate. This student incorrectly concluded that those expressions are equivalent.

When Task 2.9 was given as a free-response question, about 14% of the students arrived at an expression for the probability density that did not include the cross term. Although some students may simply have made an algebraic error, we have also found that some students explicitly stated that the cross term vanishes because the energy eigenstates are orthogonal. Figure 2.8 shows an example response in which a student reasoned that φ_1 and φ_2 are orthogonal and then arrived at an expression without the cross term.

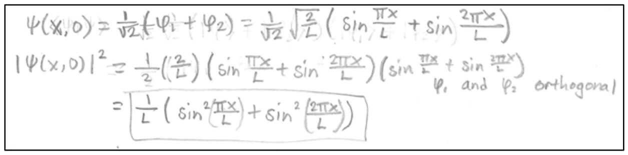


Figure 2.8. An example written response to Task 2.9.

On multiple-choice variations of Task 2.9, about one-third (31%) of all students included answer choice $\frac{1}{2}\varphi_1^2(x) + \frac{1}{2}\varphi_2^2(x)$. Although not many students gave complete explanations for why there should not be a cross term, about half of the students who made this error also stated that $\langle\psi|\psi\rangle$ could represent the probability density. These students seem to be confusing the product of energy eigenfunctions with the *inner* product between two energy eigenstates. These students appear to focus on the structures of those "products": a product between a bra and a ket, and a product between a function and the complex conjugate of another function. They do not seem to realize that the inner product (in Dirac notation) is equal to an integral (in wave function notation).

We have identified two common errors about inner products: confusing the inner product $\langle\psi|\psi\rangle$ with probability density, and not distinguishing between the product of two energy eigenfunctions and the inner product between two energy eigenstates. These similar errors together seem to suggest that many students do not differentiate between a state vector and its associated wave function. Although the difference between a state vector and its associated wave function may seem subtle, the common errors students made due to this confusion should not be overlooked.

2.5.1.c.2 Tendency to match symbols when translating inner products from Dirac to wave function notation

When we asked junior students to choose the correct integral(s) for $\langle x|\psi\rangle$ on Task 2.10, about half of the students included the incorrect integral $\int x\psi(x)dx$. Many students (29%) explained that this equation follows the "definition" of an inner product. A student response is shown below: *"We know that the inner product* $\langle a|b\rangle = \int a^* b\ dx$. *Since* x *is real,* $x^* = x$. *So* $\langle x|\psi\rangle = \int x\psi(x)dx$ *fits this form."* It seems that this student is using a template for translating expressions in Dirac notation to algebraic wave function notation. In this template, the symbols in the integrand are matched with the symbols in the bra-ket. It seems that students tend to match symbols in different notations. It is likely that they lack an interpretation of the conjugate of position eigenstate, $\langle x|$, or they incorrectly assume that the position-space wave function for a position eigenstate is *x*.

Another common (9%) incorrect response to Task 2.10 was that the *x* in the expression $\langle x|\psi\rangle$ is the position operator $\hat{x}$. Below is an example student response: "*Having the position operator act on the wave function is just the same as multiplying the two and integrating over all space, so (a) is correct*." This student appears to interpret the inner product as a position operator acts on the wave function. This inner product is considered as the multiplication between a position and the wave function. This student seems to confuse the position eigenstate with the position operator.

The results suggest that when students translate inner products from Dirac to wave function notation, they tend to match the symbols in the bra-ket with those in the integral. Some do not seem to have an interpretation of the inner product, and some interpret it incorrectly.

2.5.2 Determine Energy and Position Probabilities with Algebraic Position-space Wave Functions

Figure 2.9 shows Task 2.12 that requires students to determining energy and position probabilities using a position-space wave function. It was administered on a midterm exam in Autumn 2013 in PHYS 324. The lecture had covered the first three chapters in Griffiths textbook. Besides, students had completed a sequence of tutorials about time dependence and measurements.

On this task, students consider a position-space wave function for a particle in the infinite square well. They are asked to determine the ground state energy probability in part A and the position probability in a specified region in part B. Students were told not to evaluate integrals.

> **Task 2.12**
>
> Consider a particle in a quantum mechanical infinite square well of width a. The normalized wave function for the particle is given by $\psi(x, t = 0) = \sqrt{30/a^5}\,(x^2 - ax)$ for $0 < x < a$ at $t = 0$. Do not evaluate any integrals.
>
> **A.** Suppose you were to measure the energy of this particle at $t = 0$. Determine the probability that the energy is equal to E_1, the energy of the ground state.
>
> **B.** Suppose you were to measure the position of this particle at $t = 0$. Determine the probability that the position of the particle is measured to be between $x = 0$ and $x = a/3$.

Figure 2.9. A task requiring determining probabilities given on a midterm exam in PHYS 324

The given wave function is a superposition of energy eigenfunctions. To determine the probability of measuring the ground state energy, students need to take the inner product between ground state and the state of the particle and then take the modulus square of the inner product: $|\langle\varphi_1|\psi\rangle|^2 = \left|\int_0^a \varphi_1^*(x)\psi(x)dx\right|^2$. To determine the probability of measuring position in a region, one can take an inner product between the position eigenstate and the state of the particle, and then integrate the square of that inner product over the interested region. Since that inner product is defined as the wave function, we would get an integral of the square of wave function: $\int_0^{a/3} |\langle x|\psi\rangle|^2 dx = \int_0^{a/3} |\psi(x)|^2 dx$.

About half of the students gave a correct expression in either part A or B. Only about one-third of the students gave both correct expressions. The results suggest that students do not have a coherent model that they can use to determine probabilities for discrete and continuous cases. Data from Texas state university (TSU) and California State Polytechnic University – Pomona (CPSU) are also included. Students at TSU had completed the entire sequence of tutorials on time dependence and measurements. Students at CPSU had completed *Position, momentum, and energy measurements* tutorial, and a tutorial about energy measurements developed by the University of Colorado – Boulder. The results are very similar across all three universities, as shown in Table 2.8 and Table 2.9.

Table 2.8. Categories of student responses for the energy probability

	Correct answer*	Categories of incorrect answers				
		$\lvert c_1\rvert^2$	$30/a^5$	$\langle\psi\lvert\hat{H}\rvert\psi\rangle$	$\int_0^a \lvert\psi(x)\rvert^2 dx$	Others
UW ($N = 71$)	53%	16%	7%	6%	4%	14%
TSU ($N = 24$)	46%	N/A	N/A	4%	4%	46%**
CSPU ($N = 16$)	50%	13%	N/A	6%	13%	18%

*Some students did not have the modulus square: 1% for UW, and 13% for TSU and CSPU

**21% correct expression in Dirac notation, but incorrect integral

Although the state of the particle is represented by a wave function, many students used state vector in Dirac notation to express the energy probability. Student responses were considered as correct if they wrote a correct expression in either Dirac notation or as an integral. Student performance was very similar among all three universities. About half of the students gave a correct expression. Although uncommon with UW students (1%), a few students from TSU (13%) and CSPU (13%) did not include the modulus squared in their expression. These responses were also considered as correct. This decision was made since we considered neglecting the modulus squared a minor error compared to failing recognize that determining the probability involves an inner product.

Some students (16% for UW and 13% for CSPU) answered that the modulus square of the coefficient for the ground state wave function gave the probability. Since the wave function was not explicitly written as a linear combination of energy eigenfunctions, students need to describe how to determine the coefficient. Thus, we did not consider this type of answer correct.

Common incorrect expressions for all three universities include the expectation value of energy and the integral of wave function or probability density over the entire region. We also found that a few UW students (7%) incorrectly answered that the square of the normalization constant, $30/a^5$, gave the probability. However, we did not find students from the other universities giving this answer. About 21% of students from TSU wrote a correct expression in Dirac notation, but incorrectly translated the expression into an integral. These students seemed to have a model for determining probabilities but they did not have the fluency translating from Dirac notation into integrals. It is worthwhile to comment that students who wrote the inner product in Dirac notation without integral might also lack the fluency translating between the notations. This task did not prompt students to translate from translate from Dirac notation to integral since this prompt might have hinted students Dirac notation was needed. (Task 2.10 was used to examine student ability to translate inner products from Dirac notation to integral, see Figure 2.7.)

Table 2.9. Categories of student responses for the position probability

	Correct answer	Categories of incorrect answers		
		$\langle\psi\|\hat{x}\|\psi\rangle$	Incorrect integrands	Others
UW (N = 71)	58%	14%	14%	14%
TSU (N = 24)	58%	13%	13%	16%
CSPU (N = 16)	94%	6%	N/A	N/A

The patterns of student responses for the position probability between UW and TSU are very similar. About half of the students (58% for both UW and TSU) gave a correct expression. Common incorrect expressions include the expectation value for position, and integrals with incorrect integrands. Students from CSPU performed very well on this question. Most students (about 94%) gave a correct expression. (It is not clear to us why the prevalence is this large.) About 6% wrote the expectation value for the position.

Determining probabilities with algebraic wave functions is one of the basic requirements for students. However, many students were not able to write correct expressions for probabilities. The results suggest that many students lack a model for determining (discrete) energy probabilities and position probabilities with wave functions. Many students did not seem to recognize that taking inner products is a key step for determining probabilities.

2.5.3 Determine Energy Probability with Graphical Position-space Wave Functions

[Note that this section is reproduced and edited based on Ref [25].] Wave functions are often represented by algebraic expressions or graphs. With graphical wave functions, one can qualitatively evaluate probabilities. In this section, we are probing the extent to which students are able to use graphical wave functions to determine inner products and probabilities. We start with discussing results from three tasks administered in online surveys in section 2.5.3.a. An excerpt from individual student interviews is also presented. The purpose is to confirm our interpretation of student responses from online surveys. In section 2.5.3.b, we provide additional data from an exam to support the reliability of our data.

2.5.3.a Results from online surveys and interviews

Figure 2.10 shows threes tasks about inner products of graphical functions. Tasks 2.13 and 2.14 were given in PHYS 324, and Task 2.15 was given in PHYS 321, the first of a three-quarter sequence Electromagnetism course. All three tasks were administered in online surveys.

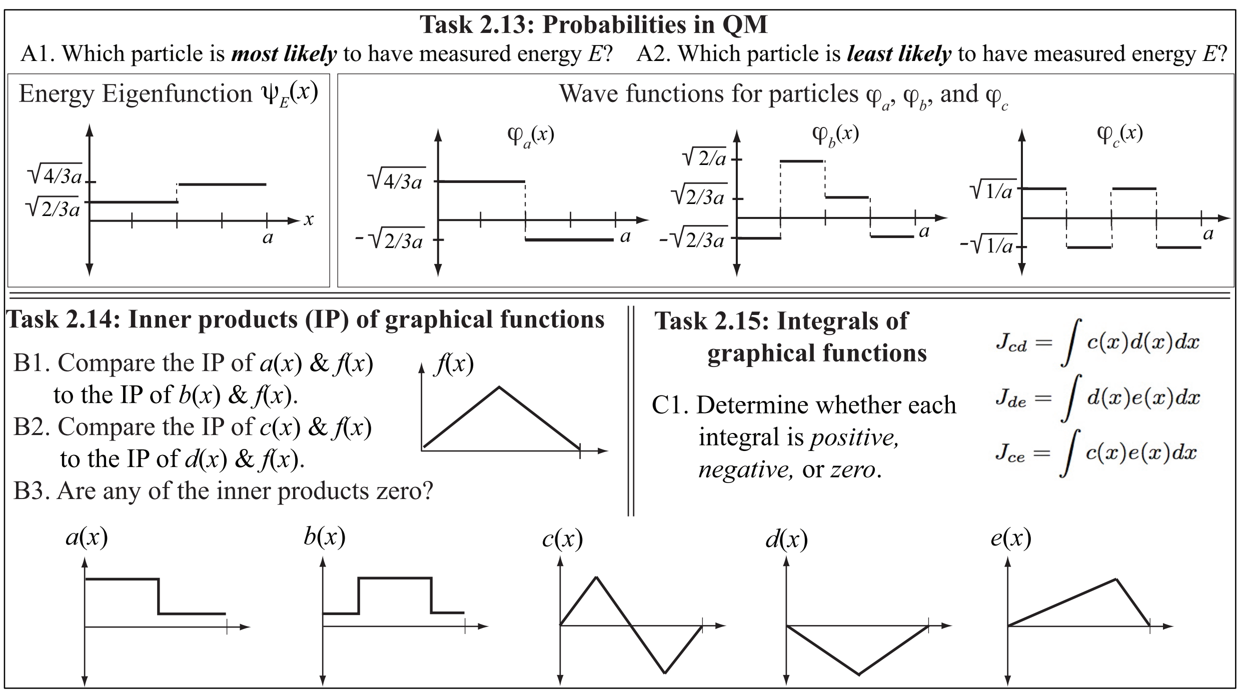


Figure 2.10. Tasks about inner products of graphical functions administered in online surveys in PHYS 324 and PHYS 321

2.5.3.a.1 Task 2.13: Probabilities in quantum mechanics

In Task 2.13, students consider four wave functions. $\psi_E(x)$ represents an energy eigenfunction with energy E of an unknown potential well. $\varphi_a(x)$, $\varphi_b(x)$, and $\varphi_c(x)$ describe three particles each in an identical potential well. Students are asked which particle is most likely to be measured to have energy E and which is least likely.

To answer these questions, one needs to evaluate the inner product between the energy eigenfunction with each of the three wave functions. Thus, the particle with a wave function that has the largest magnitude of inner product with $\psi_E(x)$ is most likely to have measured energy E. Since $\varphi_a(x)$ and $\varphi_c(x)$ both have an inner product of zero with $\psi_E(x)$, and the inner product between $\varphi_b(x)$ and $\psi_E(x)$ is non-zero, particle C is most likely.

Student performance did not seem to vary over the academic quarters, we have aggregated the data, as shown in Table 2.10. In either question, only about one-fifth of the students gave a correct answer. Based on the explanations students provided, many students did not attempt to calculate inner products. It was very common that students looked for similarities of functions. They usually compared the overall shapes or the shared values in those four regions. About 43% of the students used this type of reasoning for part A and about 41% for part B. Most of these students incorrectly concluded that particle A is most likely to be measured to have energy E since $\varphi_a(x)$ has similar shape as $\psi_E(x)$.

Table 2.10. Percentage of the students who selected each answer on Task 2.13. The correct answer for each question is bolded. ($N = 124$[a])

	Particle A	Particle B	Particle C	More than one
Most likely	53%	**22%**	7%	5%
Least likely	15%	25%	35%	**19%**

[a]PHYS 324: AUT13 and AUT14.

Task 2.13 was also used during individual student interviews ($N = 2$). The interviews were semi-structured. Students were asked to think aloud while they were thinking about the problems. An excerpt of an interview with Alan (pseudonym) is shown below:

Alan: So I am just thinking that… I think A is most likely and… Well, no, yeah, A is most likely, and then I think C is least likely.

…

Alan: I am just thinking that these are energy functions that A spends perhaps… some components of it spends time in position space has this $\sqrt{4/3a}$ value, and this negative $\sqrt{2/3a}$, which shares with this wave function here. And C does not have either of those things.

…

Interviewer: Okay. How about B?

Alan: Oh, yeah, I guess, it spends three quarters of $\sqrt{2/3a}$… which… umm… I am still leaning towards A. I guess B, I think, is somewhere in the middle.

Interview: Okay, B is somewhere in the middle because?

Alan: It shares certain similarities with ψ.

Alan, similar to other students who took the online survey, was considering the similarities between the functions. In particular, he was looking for the values that those piecewise functions shared. He seemed to argue that since $\varphi_a(x)$ shared values ($\sqrt{4/3a}$ and $\pm\sqrt{2/3a}$) with $\psi_E(x)$, particle A is the most likely; particle C is least likely since $\varphi_a(x)$ doesn't share any value with $\psi_E(x)$; and particle B is in between due to one shared value.

It is true that if two wave functions have a large magnitude (close to one) of inner product, they should look very similar. However, it is difficult to visually determine whether one pair of wave functions look more similar than another pair, especially when their magnitudes of inner products are both small. The results suggest that visual resemblance of functions is not a productive way to qualitatively compare inner products.

Correctly answering Task 2.13 requires multiple steps: recognizing that taking inner products are necessary, knowing the definition of inner products of functions, and evaluating inner products of graphical functions. It is likely that many students did not recognize the need for inner products. We have found it to be the case when the wave function was represented by an algebraic expression, as discussed in section 2.5.2. However, it is not clear from the extent to which each step is causing difficulty. The next two tasks probe whether students are able to qualitatively evaluate inner products of graphical functions.

2.5.3.a.2 Task 2.14: Inner product of graphical functions

In task 2.14, students consider several graphical functions. In the first two parts, students are asked whether the first inner product is greater than, less than, or equal to the second inner product. In the third part, they are asked whether any of the inner products is zero. The task also provides the definition of inner product of functions in words: "the inner product of two functions is the integral of the product of those functions."

Table 2.11. Percentage of the students who gave a correct answer for each question Task 2.14. ($N = 77^a$)

	B1	B2	B3
Correct answer	66%	60%	51%
And reasoning	43%	35%	47%

[a]PHYS 324: AUT14.

The results are shown in Table 2.11. About three-fifths of the students correctly compared the inner products, and about half of the students correctly found the inner product that was equal to zero. Although the overall performance seemed better compared to that on Task 2.13, it may be a result of the particular set of functions used, or it could be due to that we prompted students to compare inner products. Thus, we do not intend to compare student performance, but rather to examine student reasoning.

We have found that many students arrived at a correct answer with incorrect reasoning. About 20%-25% of the students on each question were evaluating the similarities between the functions. Many students (17%) reasoned that $f(x)$ and $d(x)$ are equal and opposite thus they canceled out. About 20% of the students answered that the inner product between $f(x)$ and $a(x)$ is the equal to the inner product between $f(x)$ and $b(x)$ because $a(x)$ and $b(x)$ have the same area underneath. Although evaluating the resemblance of functions was relatively less common on this task, this type of reasoning seemed to persist.

2.5.3.a.3 Task 2.15: Integrals of graphical functions

Instead of using the term "inner product," Task 2.15 asks students to decide whether each integral (of a product of functions) is positive, negative or zero. The functions have similar shapes as those in Task 2.14.

Table 2.12. Percentage of the students who selected each answer for each integral on Task 2.15. The correct answer for each question is bolded. ($N = 117^{a}$)

	Positive	Negative	Zero
J_{cd}	7%	21%	**71%**
J_{de}	27%	**61%**	11%
J_{ce}	25%	**56%**	16%

[a]PHYS 321: AUT14 and SPR15.

The results shown in Table 2.12 are aggregated since they did not seem to vary significantly. There were 117 who completed the task. Student performance is similar to that on Task 2.14. We found that many students did not provide detailed explanations for why selected a particular answer for each question. This may be due to the fact that the students were only asked to give an explanation after they had answered all three questions. Some students explained the generic approaches they used for the entire task, and some explained their reasoning for only one questions.

2.5.3.b Results from an exam

While we have found that most students take online surveys seriously, student explanations are not always as thorough as those for questions on exams. In order to gain more insights into the patterns of student reasoning, we analyze student responses to a task given on a midterm exam in PHYS 324. Students had completed a preliminary version of tutorials *Treating functions as* *vectors* and *Representations of wave functions* that are intended to help students with energy probabilities and inner products of graphical functions.

Task 2.16

Consider a particle in a quantum mechanical system. The wave function, ψ of the particle is shown below. The symmetric ground state wave function, ψ_1, and the anti-symmetric first excited state wave function, ψ_2, for this system are shown (both wave functions are entirely real).

Suppose the energy of this particle were measured. Is the probability that the particle is measured to have E_1, the ground state energy *greater than, less than,* or *equal to* the probability that it is measured to have E_2, the first excited state energy? If you do not have enough information to answer, state so explicitly. Explain.

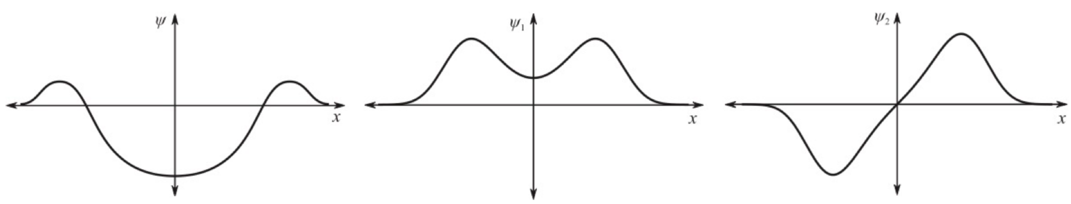


Figure 2.11. A task with graphical wave functions on a midterm exam in PHYS 324

Task 2.16 is shown in Figure 2.11. Students consider a particle in an unknown quantum potential. The wave function of the particle and the energy eigenfunctions of the ground state and first excited state are represented graphically. Students are asked whether the probability for this particle to be measured to have the ground state energy is greater than, less than, or equal to the probability that it is measured to have the first excited state energy.

To determine the probabilities, one can take the modulus squares of inner products $|\langle\psi_1|\psi\rangle|^2$ and $|\langle\psi_2|\psi\rangle|^2$. The magnitudes of inner products can be evaluated by finding the Riemann sum under the curves $\psi_1^*(x)\psi(x)$ and $\psi_2^*(x)\psi(x)$, respectively. This question can also be answered using symmetry. The wave function of the particle is a superposition of energy eigenstates and has an even symmetry. Thus, only energy eigenfunctions with even symmetry contribute. Since

$\psi_1(x)$ is even and $\psi_2(x)$ is odd, the probability of measuring E_1 must be greater than that for E_2, which is equal to zero.

About half of the students (53%) correctly answered that the probability for the ground state energy is greater. Only about 20% gave correct reasoning by evaluating the inner products and/or discussing the symmetries of the wave functions. Although uncommon (about 5%), some students recognized the need for taking inner products, but evaluated the inner products by comparing the similarities (shapes of values in some regions) between the functions. This type of responses was not considered correct.

Many students (24%) answered that there was not enough information. About 13% explained that the wave function in energy basis was needed in order to determine the energy probabilities. These students did not seem to recognize that the energy probabilities could be determined using inner products.

In order to find out the patterns of student reasoning to compare energy probabilities, we categorize the common approaches students used regardless of the answers they gave. Table 2.13 shows the categories of approaches. The categories are not mutually exclusive. Some students used more than one approach.

Table 2.13. Common approaches students used on Task 2.16. ($N = 87$[a])

Inner product	Symmetry	Discuss shapes or values of functions	Probability density
13%	17%	29%	7%

[a]PHYS 324: AUT14.

Only about 13% of the students explicitly mentioned inner product, a key step to determine probabilities. Some students (17%) argued about the symmetry, which is also a correct approach.

Many students (29%) were comparing the shapes of the functions, or the values in some regions. For example, "The graph [ψ] is more similar to ψ_1, so the probability it will have E_1 is higher," "$P_{E_1} < P_{E_2}$ since the ψ has characteristics of ψ_2 with the negative portion of the ψ." This type of approach is not productive, since the criteria of resemblance is subjective to some extent and sometimes leads to an incorrect answer. About 9% incorrectly looked for the resemblance of the probability density functions.

The results suggested that many students do not have a model for determining probabilities. They did not seem to recognize that determining probabilities involving taking inner products. It is common that students focused on the similarities in the shape or values of the functions without justification. It suggests that many students tend to use their intuitions to approach questions that requiring taking inner products of graphical functions.

2.5.4 Student Understanding of Quantum States in Vector Space

We have discussed student difficulty in determining energy probabilities when the state of the particle is described by a position-space wave function. In particular, many students do not seem to recognize an inner product is needed when the position-space wave function is not explicitly expressed as a linear combination of energy eigenfunctions. Besides, our informal observation during tutorial section suggests that students do not always recognize that a quantum state presented in a particular representation provides information about probabilities for all the observables. Therefore, we decided to examine student functional understanding of quantum states and vector spaces. In particular, we are probing whether or not students recognize that (1) a quantum state provides information about probabilities for all observables and (2) that a quantum state can be expressed in terms of the eigenstates of an observable (such as energy, position, and momentum).

We designed Tasks 2.17 and 2.18, which were given as online surveys in PHYS 324. Task 2.17 was given after about two weeks of instruction, but before students completed tutorial *Representation of wave function,* which is intended to help students recognize that any representation of a quantum state provides information about probabilities of all observables. Task 2.18 was given after students completed this tutorial.

We first discuss student performance on Task 2.17. (Note that some of the results from Task 2.17 were documented in Emigh's dissertation [66].) We then report student performance on Task 2.18 after tutorial instruction. We identify specific difficulties that persist after (lecture and) tutorial instruction, as well as additional difficulties students have with quantum states.

2.5.4.a Results from Task 2.17

Figure 2.12 shows Task 2.17, in which students consider different cases corresponding to a different representation of a quantum state: (1) a state written in terms of energy eigenstates in Dirac notation, (2) a graphical position-space wave function, (3) an algebraic expression for a position-space wave function (not explicitly in terms of the energy eigenfunctions), and (4) a generic algebraic expression for a momentum-space wave function. The online surveys displayed each case one at a time. Students were not permitted to revise their answers to the previous cases. In each case, students were asked to decide whether it is possible to determine the probabilities for energy, position, and momentum measurements.

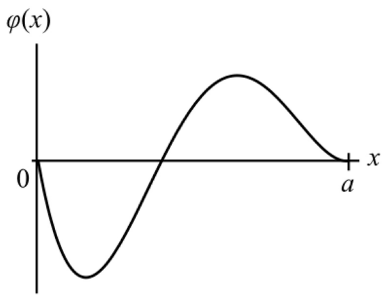
The results are shown in Table 2.14. The data are aggregated. Student responses were considered correct if they either answered that it was possible to find all those probabilities, or if they argued that the actual wave function was needed in order to find the probabilities. Only about 20%-30% of the students correctly answered in each case that it was possible to determine the probabilities for energy, position, and momentum measurements.

The most common incorrect answer was that only the probability for one observable could be determined: the one that corresponds to the basis in which the state is represented in. For example, in case 1 (the energy basis), about 30% of the students answered only the energy probability could be determined. They explained that the position and momentum probabilities could not be determined because the state is written in the energy basis. Similarly, about 30% and 20% in cases 2 and 3, respectively, answered that only the position probability could be determined.

Task 2.17

Case 1: Consider a particle in the infinite square well. Suppose this particle is described by the following expression in Dirac notation: $|\psi\rangle = \frac{3}{5}|\psi_2\rangle + \frac{4}{5}|\psi_3\rangle$.

Case 2: Consider a different particle in the infinite square well. This particle is known to have the following wave function shown below.



Case 3: Consider a different particle in the infinite square well. This particle is known to have the wave function given by $\psi_A(x) = N(\sin(\pi x/a) + \sin^3(\pi x/a))$, where N is a constant.

Case 4: Consider a different particle in the infinite square well. This particle is known to have the wave function in momentum space $\psi_B(p)$. (The exact expression is not shown.)

For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Quantity 1: The probability that an energy measurement will result in E_2, the first excited-state energy.

Quantity 2: The probability that a position measurement will find the particle on the left half of the well.

Quantity 3: The probability that a momentum measurement will find the particle moving to the left.

Please explain your reasoning for all of the above cases.

Figure 2.12. A sequence of questions[1] about probabilities administered in online surveys in PHYS 324

It was very common that students reasoned that the state in a particular representation does not provide information about other observables. A student explained that "*The probability of taking an energy measurement is just a simple inner product between the energy state and psi. Energy states tells you nothing about position or the direction a particle is moving*." This student

[1] The results from responses to the first three cases were documented in Emigh's dissertation [66].

correctly recognized that the energy probabilities can be determined using the inner products. However, they incorrectly assumed that there was no information about the position or momentum. Another student reasoned that "*Here we have no idea which energy eigenstate or combination it is in, we cannot tell the probability of that eigenstate. For position, we simply square it and integrate over the desired length. Again, we have no information about direction of motion here*." Similarly, this student reasoned that we could not tell what combination of energy eigenstates it is when the state was described by a graphical position-space wave function.

Table 2.14. The percentage of correct answer to all the questions in each case on Task 2.17.

	Case 1	Case 2	Case 3	Case 4
Correct answer	32% (N = 226[a])	26% (N = 145[b])	29% (N = 226[a])	20% (N = 81[c])

[a]PHYS 324: AUT14, AUT15, and AUT16. [a]PHYS 324: AUT14 and AUT15. [c]PHYS 324: AUT16.

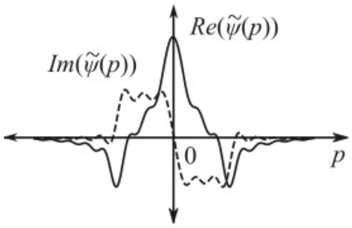
2.5.4.b Results from Task 2.18

Figure 2.13 shows Task 2.18, which was given after students completed tutorial *Representation of wave function tutorial.* Question 1 is a modified version of case 4 in Task 2.17. Instead of a generic algebraic momentum-space wave function, students consider a graphical momentum-space wave function. They are asked to decide whether it is possible to determine the probabilities for energy, position and momentum measurements.

In question 2, students consider a generic state vector. They are asked to decide whether it is possible to express the state vector in terms of energy eigenstates, position eigenstates, and momentum eigenstates, respectively, in each question.

Task 2.18

Question 1: Consider a particle in the infinite square well. Suppose this particle is described by the wave function in momentum space as shown.



For each of the following quantities, state whether or not it is possible to determine the quantity of interest given the above representation of the state.

Quantity 1: The probability that an energy measurement will result in E_2, the first excited-state energy.

Quantity 2: The probability that a position measurement will find the particle on the left half of the well.

Quantity 3: The probability that a momentum measurement will find the particle moving to the left.

Please explain your reasoning for all of the above cases.

Question 2: Consider a particle in the infinite square well. The particle is described by the state vector $|\psi\rangle$.

A. Is it possible to express $|\psi\rangle$ in terms of energy eigenstates $|\psi_n\rangle$ of the system? Explain.

B. Is it possible to express $|\psi\rangle$ in terms of position eigenstates $|x\rangle$? Explain.

C. Is it possible to express $|\psi\rangle$ in terms of momentum eigenstates $|p\rangle$? Explain.

Figure 2.13. A sequence of questions about vector space administered in online surveys in PHYS 324

The results are shown in Table 2.15. On question 1, about 39% of the students gave a correct answer. They either answered that it was possible to determine all three probabilities or argued that the actual wave function was needed in order to change basis. Student performance seemed improved after tutorial instruction. However, more than half of the students still did not recognize that all (energy, position, and momentum) probabilities are possible to be determined. About 22% (of all students) reasoned that the state is in the momentum basis thus neither energy probability nor position probability can be determined. Others either answered that energy probability cannot be determined or answered that position probability cannot be determined.

On question 2, about 70% of the students correctly answered that the state vector could be expressed in terms of any of those eigenstates (energy, position, and momentum eigenstates). However, only about 20% correctly reasoned that the eigenstates form a complete basis or the coefficients can be determined using inner products.

Table 2.15. Student performance on Task 2.18.

	Question 1	Question 2
Correct answer (N = 64[a])	39%	70%

[a]PHYS 324: AUT15.

About one-quarter of all students regarded expressing $|\psi\rangle$ in terms of $|x\rangle$ as writing the position-space wave function $\psi(x)$. For example, one student answered that *"We can get to position eigenstates from energy eigenstates by expressing the energy eigenstates as functions of x."* It seems that this student started with $|\psi\rangle$ written in terms of $|\psi_n\rangle$. That is, $|\psi\rangle = \sum c_n |\psi_n\rangle$. The student then replaced $|\psi_n\rangle$ with $\psi_n(x)$, and thus $\sum c_n |\psi_n\rangle$ became $\sum c_n \psi_n(x)$. It appears that this student regarded "expressing $|\psi\rangle$ in terms of $|x\rangle$" as "expressing $|\psi\rangle$ as a function of x." It is likely that the student did not differentiate between a position eigenvalue x and a position eigenstate $|x\rangle$.

Another student responded that *"Represent the function as a continuous function of positions. The position eigenstates are all positions."* It appears that this student called the state vector $|\psi\rangle$ "the function" and represented the state vector as a continuous function of position, which is $\psi(x)$.

Similarly, this student also seemed to regard "expressing $|\psi\rangle$ in terms of $|x\rangle$" as "expressing $|\psi\rangle$ as a function of x."

It is also common that students regarded expressing $|\psi\rangle$ in terms of $|p\rangle$ as writing the momentum-space wave function $\psi(p)$. Many explained explicitly or implicitly that one can use Fourier transform to express $|\psi\rangle$ in terms of $|p\rangle$. For example, one student wrote, *"It's possible to convert a position expression of the wave function into an expression in momentum space. Doing so will put the wave function into terms of momentum eigenstates."* It is correct that one can transform a position-space wave function to a momentum-space wave function. However, this student seemed to believe that the momentum-space wave function would be the expression for $|\psi\rangle$ written in terms of $|p\rangle$. It seems that this student also refers the state vector as "the wave function", since they said "put the wave function into terms of momentum eigenstates" rather than "put the state vector in terms of momentum eigenstates."

It seems that many students do not differentiate between a state vector $|\psi\rangle$ and its associated position-space (or momentum-space) wave function $\psi(x)$ (or $\psi(p)$). They seemed to believe that these expressions are all equivalent. Some students also do not distinguish between a position eigenvalue x and a position eigenstate $|x\rangle$. This is consistent with what we found in section 2.5.1.

Student performances on question 1 and question 2 do not seem to be consistent. Most (70%) of the students seemed to recognize that one can write $|\psi\rangle$ in terms of the eigenstates of different observables, even if many do not differentiate between a state vector $|\psi\rangle$ and its associated wave function $\psi(x)$, or $\psi(p)$. However, only 39% recognized that one can determine energy or position probability when given the momentum-space wave function. The results suggest that although students recognize that one can represent a generic state vector in a particular representation (energy, position, or momentum), many do not seem to know how to translate between different representations and determine probabilities. Being able to translate between different representations and determine probabilities requires students to understand how a generic state vector is related to each representation. Thus, it is important that students recognize that a position-space wave function is an inner product between the position eigenstate and the state vector.

2.6 Probing Student Reasoning From A Different Perspective: How Different Notations Can Support And/Or Hinder Student Reasoning

Our quantitative study discussed in the previous sections suggest that many students have difficulties with translating expressions between different notations and with determining probabilities. Thus, we conducted a follow-up qualitative study to prove how different notations may impact student performance. In this section, we analyze student responses to interview tasks using the structural features of quantum notation framework [33]. This framework allows us to discuss how the various structural features of Dirac notation and of wave function notation may support and/or hinder students' reasoning.

Individual interviews ($N = 6$) were conducted near the end of the second quarter of the junior-level quantum course after relevant lecture instruction and also after students had worked through early versions of tutorials intended to improve student understanding of the quantum states and different representations. The interviewees were all the students who volunteered to participate in this study. The names included in this paper are pseudonyms.

The interviews were semi-structured and used a think-aloud protocol. Each consisted of two parts. In part 1, students were given Task 2.12 as shown in Figure 2.9. (None of the participants came from the population who had Task 1 on a midterm exam.) After the students had worked through the task, they were asked to compare the approaches they used to determine the probabilities for energy and position measurements. Students were prompted to discuss similarities and differences between the two questions. In part 2, students were given expressions related to quantum states and inner products (*e.g.,* $|\psi\rangle$, $\psi(x)$, and $\langle x|\psi\rangle$) one at a time. They were asked to explain what each expression represents and how they make sense of them. The interviewer also asked students to translate the inner products from Dirac to wave function notation.

The purpose of these open-ended questions was to gain more insight into how students reason about state vectors and inner products.

2.6.1 Theoretical Framework: Structural Features of Quantum Notations

Different representations are commonly used in physics. Quantum mechanics, in particular, involves functions, graphs, matrices, and Dirac notation. There is evidence from prior research that student performance can be influenced by the representations that are used [33,67]. Heckler and Scaife, for example, found that student performance on adding and subtracting vectors in (algebraic) *ijk* format is significantly better than that in (graphical) arrow format [67]. Gire and Price [33] probed the extent to which different notations impact student performance in quantum mechanics. They conducted interviews with students who were asked to use multiple notations to represent a superposition state and then to compute the expectation value of energy.

Gire and Price focused on three notational systems: Dirac notation, algebraic wave function notation, and matrix notation. We do not discuss matrix notation because it is not commonly used for the infinite square well potential and no students used this notation spontaneously. Their observations and analysis suggested four characteristics of the notations that might play a role in student performance. They called these *individuation, externalization, compactness,* and *symbolic support for computation.* These are defined below.

Individuation is "the degree to which important features [basis states] are represented as separate and elemental." [33] In Dirac notation, a quantum state written in terms of basis states (e.g., $a|\varphi_1\rangle + b|\varphi_2\rangle$) can be thought of as having a high degree of individuation since the basis states are represented by individual kets and the probability amplitudes can be easily identified. In contrast, in wave function notation, a linear combination of eigenstates is usually compressed into an expression where the basis states are not visible. For example, a superposition of energy eigenstates for a particle in the infinite square well may be expressed as a polynomial, e.g., $\psi(x) = \sqrt{30/a^5}\,(x^2 - ax)$. In this case, the wave function notation is less individuated. Since the probability amplitudes for the energy eigenstates are invisible, this notation provides little help for students to determine the probabilities for energy.

Externalization "describes the degree to which elements and features are externalized with markings included in the representation." [33] Wave function notation can be regarded as relatively highly externalized, since it makes explicit the detailed mathematical behavior of the quantum states. For example, the wave function $\sqrt{30/a^5}\,(x^2 - ax)$ is a polynomial and the energy eigenfunctions for the infinite square well, $\sqrt{2/a}\,\sin\left(\frac{n\pi x}{a}\right)$, are sinusoidal. With these functions, one can directly perform a variety of mathematical operations. In contrast, the externalization of Dirac notation is low. Quantum states are represented by kets and the mathematical behavior is implicit. A superposition of energy eigenstates in Dirac notation, $|\psi\rangle = a|\varphi_1\rangle + b|\varphi_2\rangle$, does not explicitly show the mathematical behavior of the state.

Compactness is "a measure of how much space and writing is needed." [33] The relative compactness of two notations may vary. For example, in Dirac notation, a generic state vector $|\psi\rangle$ or a basis state represented by a single ket (*e.g.,* an energy eigenstate is $|\varphi_n\rangle$), has higher compactness than the same state in wave function notation (*e.g.,* an energy eigenfunction for the infinite square well is $\sqrt{2/a}\,\sin\left(\frac{n\pi x}{a}\right)$). In contrast, a superposition state in wave function notation may have high compactness, if the expression is algebraically simplified. For example, $\psi(x) = \sqrt{30/a^5}\,(x^2 - ax)$, a polynomial with only two terms, contains an infinite number of superimposed energy eigenstates. Moreover, a superposition state with many non-zero expansion coefficients in Dirac notation may have low compactness if the coefficients are expressed numerically, or high compactness if the coefficients are specified generically (*i.e.,* as $\sum_n c_n|\varphi_n\rangle$).

Symbolic support for computation measures the degree to which the physical shape of a notation system supports computation. Dirac notation is considered to have a high degree of symbolic support. The bras and kets have asymmetric shapes: a straight vertical line on one side and an angle bracket on the other side: $| \quad \rangle$ or $\langle \quad |$. In this notation, it is clear (to those who have experience with Dirac notation) that any expression with an unclosed bracket is a vector in Hilbert space. The asymmetry of unclosed brackets provides visual cues for the different sequences of elements in different operations: inner products and outer products. An inner product is denoted by a closed bracket, $\langle \quad | \quad \rangle$, resulting in a scalar, whereas an outer product has a different sequence of elements, $| \quad \rangle\langle \quad |$, resulting in an operator. In contrast, the wave function notation provides little symbolic support for computation. Gire and Price gave an example in which a student immediately wrote down Dirac notation $\langle\psi|\hat{H}|\psi\rangle$ when asked to calculate the expectation value of the energy. When the interviewer prompted the student to use wave function notation, the student admitted that he did not remember how to compute in that notation. After the interviewer hinted that there were integrals involved, the student incorrectly wrote $\int \psi\psi^{*}\hat{H}dx$. Overall, the interviewees tended to make more errors with the order of elements when they were using wave function notation.

2.6.2 Analysis and Results

The interviews were audio taped and transcribed. Transcripts were coded based on the four structural features categorized in Ref. [33]. We select and present examples that are insightful to demonstrate in detail how the structural features may support and/or hinder student sense-making about inner products. We do not attempt to claim that the example excerpts are representative and generalizable to larger populations. The goal is gain insights into potential strategies that can be used to address these common errors students make when they reason about inner products in different notations.

Our analysis suggests that different structural features can be regarded as facilitating and/or impeding student sense making about inner products. The primary findings are categorized and discussed below.

2.6.2.a High symbolic support of Dirac notation may facilitate student sense-making of inner products

When Amy was working on Task 2.12 during an interview, she correctly stated that the wave function could be written as a linear combination of energy eigenfunctions. She also referenced the Born rule and argued that the modulus square of the first coefficient gives the probability for the ground state energy. However, she was not sure how to determine the coefficient. At first, she tried acting with the Hamiltonian on the wave function. She rewrote the Hamiltonian in terms of the momentum and position operators and worked through the algorithmic procedures. Later, she realized that what she needed was a Fourier series.

Amy: *So I think… you need a Fourier or sine series in order to get… I probably should have done a Fourier transform on my thing right away, because I could get my ψ in terms of an infinite sum of things, infinite sum of energy eigenfunctions. I mean this was revealing, but I don't think that it actually helped me. I did bring to conclusion that I do need Fourier transform.*

Amy realized that she was on the right track but failed to recall how to determine the Fourier series. Although she used the term "Fourier transform," it is likely she meant Fourier series based on the rest of her interview. The interviewer suggested moving on. After working through the entire Task 2.12, the interviewer gave Amy several expressions, such as $|\psi\rangle$, $\psi(x)$, and $\langle\varphi|\psi\rangle$, one at a time. When asked to make sense of $\langle\varphi|\psi\rangle$, Amy said that she thought about inner products as dot products and interpreted them as the projections between spatial vectors.

Amy: *Actually looking back, I think this is the idea of Fourier transform. All the sine(s) [energy eigenfunctions] are orthogonal, right? You do this dot product and everything cancels out except for the particular value you are looking for.*

Sense-making of the inner product $\langle\varphi|\psi\rangle$ appears to have produced an "aha" moment for Amy, who then realized how to determine the coefficient in Task 1A. She realized that the inner product between the ground state and the given state vector is one of the coefficients in a Fourier series. She wrote down the inner product in Dirac notation and then immediately translated it into wave function notation as an integral. The high symbolic support of Dirac notation appears to have helped Amy reason about inner products by forming an analogy to dot products between spatial vectors. Once she realized an inner product was needed, she was able to translate it from Dirac to wave function notation immediately.

Our results suggest that the high symbolic support of Dirac notation may help students determine the probability for an energy measurement when the wave function is not written as a linear combination of energy eigenfunctions. In particular, Dirac notation can be considered as providing strong visual cues to the interpretation of inner products. In Dirac notation, an unclosed bracket, $|\quad\rangle$ or $\langle\quad|$, represents a vector in Hilbert space, and a closed bracket, $\langle\quad|\quad\rangle$, represents a scalar. Thus, an expansion coefficient is picked out as a result of the closed bracket. The physical shape of the "bracket" can support students' ability to make a connection between the inner product and the probability by interpreting an inner product as the projection or overlap between two state vectors. Using Dirac notation as a template seems a productive strategy for determining energy probabilities.

2.6.2.b Low individuation of wave function notation may hinder student ability to find expansion coefficients

Another student, Jack, stated that the ground state wave function in Task 1 would be the x term in the polynomial.

Jack: *But I am used to seeing the ψ functions like trig or something, that's more… when you have the form like [sin]* $\frac{n\pi x}{a}$*, that makes more sense to me when it's asking about the ground state, so I don't actually know right now or I'm going to commit, I guess the more appropriate answer... But it seems to me that the ground state of this function would be when you have the single… you have the* $x^2 - ax$*. I am thinking the a is the determining factor, or the x is the value that would yield… you know it's like a superposition of states. That's the word I should, I am looking for. The energy* E_1 *is what you get when you find the particle in the state defined by the single, you know x instead of* x^2*, so the probability would be based off the factor a and then the coefficient in front of the whole thing.*

Jack correctly recalled that the energy eigenfunctions for the infinite square well are sine functions. However, he did not recognize the expansion coefficients can be determined by integration. This seems to be consistent with the low symbolic support of wave function notation, which provides little visual cue on what mathematical operation needs to be performed. Moreover, since the given wave function is not expressed in terms of the sine functions, both the energy eigenfunctions and their associated probability amplitudes are invisible. The appearance of the polynomial, a combination of two terms, may have led Jack to state instead that the x term corresponds to the ground state, and the x^2 term corresponds to the first excited state, as shown in Figure 2.14. Then, he incorrectly concluded that the coefficient for the ground state is just the coefficient in front of function x and the probability of measuring the ground state energy is the square of that coefficient. The low individuation of wave function notation seems to have had a negative impact on Jack's reasoning.

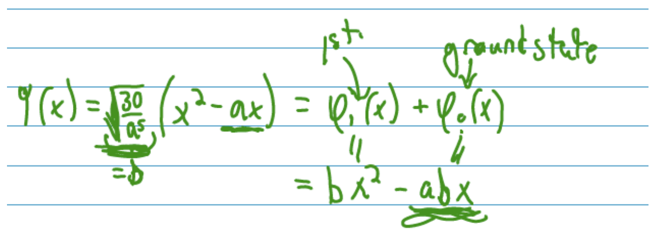


Figure 2.14. Jack's written response during an interview

The results seem to suggest that wave function notation is not a productive notation for determining expansion coefficients. This may result from the low individuation and low symbolic support of wave function notation, especially when the wave function is not expressed as a linear combination of energy eigenfunctions. When it is instead compressed into a polynomial, the energy eigenfunctions and their associated coefficients are highly encoded. Compared to Dirac notation, wave function notation provides little visual cue on how the coefficients can be determined. Although experts may recognize an integral is needed to determine the coefficient, students do not seem to have developed this expertise.

2.6.2.c High compactness of Dirac notation may impede student ability to distinguish probability density from an inner product

Although struggling for a while at the beginning, student Nathan correctly answered both parts in Task 2.13. When asked whether the procedures he used to solve the two parts are similar or different, Nathan stated that they were different to him. He explained that in part a he took the "inner product of the state against the wave function" and in part b he took "an integral of probability density over the region of interest instead of the entire well."

Int.: So you do not take inner product in part b like you did you part a?

Nathan: *There's… I guess there's an inner product. I guess, this term [He points to* $\psi^*\psi$*], right there, is the inner product of the state with itself [He writes* $\langle\psi|\psi\rangle$*]. Yeah, Yeah! So it is an inner product I suppose. But in [part] a, inner product is sort of where it sucks out the value you're looking for coz it has energy eigenstate, the, the eigenvalue, yeah the eigenstate you are interested in, and then integrate over the whole well. Then the second one, you just inner product the state with itself and then look for specific region coz you want the position, probability of position.*

Int.: *So what is the inner product of the state vector with itself? What is the result?*

Nathan: *Would be* $\psi^*\psi$*, which is real. Would be … [He writes* $30/a^5\,(x^4 - 2ax^3 + a^2x^2)$*]*

Nathan did not seem to recognize that the wave function is an inner product. Instead, he claimed that the probability density is an inner product. When asked for the result of the inner product of the state vector with itself, he did not invoke an integral, but rather calculated the probability density.

An inner product in Dirac notation is highly compact. When the inner product $\langle\psi|\psi\rangle$ is translated into wave function notation, it is expressed as an integral $\int_0^a \psi^*(x)\psi(x)dx$. Nathan did not seem to realize that the inner product (in Dirac notation) is equal to an integral (in wave function notation). The compactness of the inner product in Dirac notation conceals a highly relevant mathematical operation (the integral) of an inner product in wave function notation. The compactness of Dirac notation appears to impede Nathan's ability to translate the inner product from Dirac to wave function notation.

2.6.2.d Low externalization of Dirac notation may hinder student ability to interpret quantum states

When student Steve was determining the position probability on Task 2.12b, he spontaneously wrote $\langle x|\psi\rangle = \int x\psi(x)dx$.

Int.: *If we break down the inner product… the thing on the left… what would you call that?*

Steve: *The operator. It is the thing measuring ψ. I would technically put x hat [$\hat{x}$], but x hat is the same as x, which in that case, it's like what's the point of that? It makes more sense if we have p hat [$\hat{p}$]. p hat is $-i\hbar$ d/dx. This is not clearly… It's not like a regular mathematical function that you can just put in there. It has to be operating upon something, like your Hamiltonian kind of thing. Because we can have operator of position, momentum, Hamiltonian as our energy operator.*

It seems that Steve associates the inner product with a measurement and with an operator. He incorrectly externalizes the abstract symbol $\langle x|$ to something that he is familiar with: the position operator. Steve also compares it to a momentum operator to support his argument. Steve's interpretation of $\langle x|$ appears to be based only on the letter symbol that is used inside the bra rather than the entire bra notation. He relates that symbol to a physical operation, a measurement, that is acted on the given state vector $|\psi\rangle$. Besides, he does not seem to differentiate between a measurement and an operator. It appears that the low externalization of Dirac notation hinders Steve's ability to sense make about inner product $\langle x|\psi\rangle$.

On the other hand, another student Tyson, seemed to interpret $\langle x|$ as a state during an interview. "*So the way you would write it as an integral if you did, was like the state x times ψ dx.*" Then he wrote $\langle x|\psi\rangle = \int x^*\psi(x)dx$, where x^* represents the complex conjugate of x. It appears that Tyson recognized the bra $\langle x|$ as a state since he used the phrase "state *x*." However, he argued that this inner product could not be written as an integral.

Tyson: *Yeah, there's no like state x.*

Int.: *There's no like state x?*

Tyson: *No, x should be a position, right? It's not a state.*

Int.: *It's not a state. Then what does that… the half of the expression has a… It's a bra x, right? What does that mean?*

Tyson: *The bra part of it? That would indicate a complex conjugate or Hermitian conjugate.*

Int.: *Okay. What about a ket x? (The interviewer wrote down* $|x\rangle$*) What does that represent?*

Tyson: *Just x.*

Int.: *Just x. You mean this x? (The interviewer wrote down x.) They are the same?*

Tyson: *Umm... Yeah as far as I… That's what I was inclined to answer.*

Int.: *Okay. It's the same as function x, so you don't call this a state. That's not a state.*

Tyson: *No, it shouldn't be.*

Tyson correctly recognized that an unclosed bracket should be a state and function *x* does not represent a state (at least not a state that is commonly used in a junior-level textbook). However, he incorrectly believed that $|x\rangle$ is equal to function *x* (which represents a position). Thus, he concluded that this inner product could not be written as an integral because function *x* does not describe a state.

It is perhaps not surprising that students attempt to externalize $|x\rangle$ to function *x* since they share the same symbol. Although Steve and Tyson use different lines of reasoning, they both appear to externalize $\langle x|$ based on the symbol x used in the bra. They both seem to confuse x as a function and x as a *label* either for the position operator or for an eigenstate of that operator. Again, the high compactness and low externalization of a state vector in Dirac notation may impede student ability to interpret quantum states during translation between notations.

We have evidence showing that students not only tend to externalize quantum states incorrectly, but also that they do not differentiate between a quantum state and a wave function denoted by the same letter symbol. Four out of five interviewees (one student was not asked this question) stated that they would call $|\psi\rangle$ a wave function (often expressed as $\psi(x)$). Below is an example from Nathan's interview:

Int.: *What does this expression [$|\psi\rangle$] represent?*

Nathan: *It [$|\psi\rangle$] represents wave function of some kind of ket form.*

Int.: *Is it similar to or different from $\psi(x)$?*

Nathan: *I think they [$|\psi\rangle$ and $\psi(x)$] are the same.*

Nathan was able to externalize the state vector $|\psi\rangle$ to its wave function. However, he incorrectly believed that $|\psi\rangle$ is also a wave function just written in a different notation. It is very likely due to that both expressions are denoted by the same letter symbol ψ. This seems to be consistent with Tyson's belief that $|x\rangle$ is equal to x. The results suggest that many students do not always differentiate between state vectors and wave functions. This may be a reason why students do not always recognize that $\psi(x) = \langle x|\psi\rangle$, a key component required for recognizing that the procedure for determining position probabilities are analogous to that for discrete cases.

2.6.3 Summary

We have found that Dirac notation, due to its high symbolic support, can be helpful for students to determine (discrete) energy probabilities using inner products. The strong visual cue provided by the bracket $\langle \quad | \quad \rangle$ supports student sense-making of inner products and their relations to probabilities. Wave function notation, however, does not provide a similar level of support for determining energy probabilities. We argue that using Dirac notation as a template may be a productive strategy for students to make sense of the Born rule. The results from interviews are consistent with what we found from student written work: many students spontaneously use Dirac notation for determining probabilities, even if the state of the particle is represented in wave function notation.

However, we have also found that the high compactness and low externalization of Dirac notation may hinder student ability to translate inner products from Dirac to wave function notation. Inner products in Dirac notation are highly compact. An integral is encoded in the "bracket" that represents an inner product. This seems to be correlated with the arise of confusion between the inner product $\langle\psi|\psi\rangle$ and probability density. Moreover, the externalization of Dirac notation is low and the letter symbol used in the unclosed bracket does not necessary represent the wave function associated with that state vector. However, some students tend to incorrectly externalize state vector $|x\rangle$ to function x or the position operator.

The results about the high compactness and low externalization of Dirac notation appear to be consistent with several common errors we have identified in the quantitative study discussed in the previous sections: many students (1) believe that $\langle\psi|\psi\rangle$ is equal to $\psi^*(x)\psi(x)$, (2) think $\varphi_1(x)\varphi_2(x)$ equals to zero due to the orthorgonality of energy eigenstates, and (3) tend to match the symbols in the "bracket" to those in the integrals. These errors seem to suggest that many students are unable to differentiate between a state vector and its corresponding wave function. In addition, we have found that many students expressed equivalence between state vectors and wave functions during individual interviews. They stated that they would call both $|\psi\rangle$ and $\psi(x)$ a wave function.

2.7 Summary

This chapter has discussed an investigation into student understanding of the use of inner products for determining quantum probabilities. The investigation made use of both quantitative and qualitative methods. The results suggest that many students do not have a model that they can use to determine probabilities for discrete and continuous cases. Students do not always recognize that determining probabilities involves taking inner products. Moreover, many students do not differentiate between a state vector and its associated wave function. This confusion seems to be widespread and to hinder student ability to determining probabilities.

We found that many sophomore-level students lack the fluency of basic computations involving inner products and probabilities. In particular, we identified several common errors students make. Many students treat state vectors like spatial vectors without taking the complex conjugate, especially when asked to determine the magnitudes of states vectors or find orthogonal state vectors. This type of error is less common when students are directly calculating inner products. It may be due to that the terms “magnitudes” and “orthogonal” reinforce the geometrical interpretations for spatial vectors so that students are more likely to treat state vectors as spatial vectors. Similarly, students tend to treat a modulus square as a “regular” square without taking the complex conjugate on tasks that require them to determine probabilities. Other common errors involve neglecting to change basis or making basic algebraic errors.

We have also found that students do not always recognize that determining probabilities involves taking inner products. For example, when inferring a quantum state based on statistical measurement results, many students confuse the probabilities with the expansion coefficients (which can be determined from the inner products). Many have difficulty in determining probabilities for angular momentum when the state vector is not written in terms of the basis states of interest (e.g., on a question that asks for the probability of the x-component when the state is expressed in the J_z basis). We have found that many students do not recognize that determining energy probabilities involves taking inner products when the quantum states are described by graphical or algebraic position-space wave functions (that are not explicitly written in terms of the energy eigenfunctions). On the questions involving graphical wave functions, students tend to use visual resemblance of graphical functions. For example, students tend to say ψ_A has a greater probability of finding energy E since ψ_A looks more similar to the energy eigenfunction than ψ_B does. These results suggest that students do not have a model that they can use to determine probabilities.

A further investigation suggested that many students do not recognize that any representation of a quantum state provides information about the probabilities for all observables. They often incorrectly state that only the probability for one observable can be determined; namely the one that corresponds to the basis in which the state is represented. These results seem to explain why students have difficulty in determining probabilities when the quantum state is not expressed in the basis of interest.

Finally, we found that many students do not recognize that a wave function is an inner product. Many confuse a wave function with its associated state vector. This confusion seems to impede student ability to translate expressions between notations and to build a model they can use to determine probabilities for discrete and continuous cases. We conducted some interviews to probe student thinking in greater detail and used the *structural features of quantum notation* framework [33] to interpret our data. The results suggest that the interviews seem to be consistent with the finding from the written tasks.

Chapter 3. Development and Assessment of Tutorials on Quantum States and Inner Products

Chapter 2 presents results from an investigation into student understanding of the use of inner products for determining probabilities. The results suggest that students often do not recognize that determining probabilities involves inner products; many students do not seem to recognize that a quantum state expressed in a particular representation encodes information about the probabilities for all possible observables; It is very common that students do not distinguish between a state vector (in Dirac notation) and the corresponding wave function, which is an inner product between a position eigenstate and the state vector. The confusion between a state vector and its corresponding wave function appears to impede student ability to determine probabilities coherently for discrete and continuous cases.

In order to address these difficulties, we have developed a new tutorial *Probability amplitude* intended for use in PHYS 324. We have also modified three existing tutorials, *Dirac notation*, *Treating functions as vectors*, and *Representations of wave functions*. *Dirac notation* is used in PHYS 225, while the other two are used in PHYS 324.

In section 3.1, we discuss modifications of the tutorial *Dirac notation*. In section 3.2, we discuss the initial development of the tutorial *Probability amplitude*, and the assessments of this tutorial and the new sequence of tutorials. Finally, we discuss additional modifications and assessment of these tutorials in section 3.3.

3.1 Modification of Tutorial *Dirac Notation*

The development of the early versions of tutorial *Dirac notation* tutorial is documented in the PhD dissertation of Emigh [66]. A primary objective of the early versions of this tutorial is to help sophomore-level students gain familiarity with state vectors written in Dirac notation, which is often considered highly compact but quite abstract.

In this section, we briefly describe the original tutorial *Dirac notation*. We then describe several changes that have been made to the tutorial as a result of the findings discussed in chapter 2.

3.1.1 Original Tutorial *Dirac Notation*

The original version of the tutorial (see Appendix A, Figure A.1) begins with a scenario that involves spatial vectors in 2-dimensional Cartesian coordinates. Students are asked to write mathematical expressions for the vectors shown in a figure. Then they are asked to evaluate and interpret inner products (dot products) in several cases. Lastly, students consider a n-dimensional space. They are asked to generalize their expressions for inner products to the case of n-dimensional vectors.

In the second section of the tutorial, students consider a 6-dimensional Hilbert space. The aim of this section is to help students draw a comparison between state vectors and spatial vectors. Students are prompted to discuss the similarities and differences between inner products for state vectors and dot products for spatial vectors. In particular, students are asked to calculate the magnitudes for state vectors. Besides, students are given incorrect calculations for inner products in which the dual state is not complex conjugate, and asked to identify the errors.

The third section is in the context of spin-1/2 states. It is intended to help students with changing basis. Students are given a state vector written in a particular basis (e.g., the S_z or S_x basis), then they are asked to find the inner products between the state vector and different basis states (e.g., $|+\rangle_z$ or $|+\rangle_x$). Students are guided to recognize that some inner products are easier to compute in a particular basis. Lastly, they are asked to rewrite the state vector in a different basis.

3.1.2 Modified Tutorial *Dirac Notation*

As discussed in chapter 2, many students treated state vectors as spatial vectors, failing to take complex conjugates, after lecture and tutorial instruction. Some students treated state vectors written in different basis as if they are in the same basis. The results motivated us to modify the tutorial *Dirac notation*. The modified tutorial (see Appendix A, Figure A.2) is intended to guide students to relate mathematical expressions to physical quantities and experimental implications.

In the original tutorial, students consider a 6-dimensional Hilbert space. This was removed in the modified tutorial. We restrict the scenario to a spin-1/2 system. The benefit of having a spin-1/2 system is two-folded: (1) the mathematical computation is relatively straight forward (compared to a 6-dimensional space) so that students can focus on the concepts, and (2) students have seen various Stern-Gerlach experiments in the lecture and textbook so that they can be guided to relate the mathematical expressions to Stern-Gerlach experimental results.

To focus on the comparison between 2-dimensional spatial vectors and spin state vectors, we did not ask students to generalize their answers about dot products to the case of *n*-dimensional vectors in first section. This change allows students to dive into a quantum scenario quickly.

We have made substantial changes to the second section. The primary goals of the second section are to help students recognize (1) the need to take complex conjugates when taking inner products and modulus squares, and (2) the need to normalize quantum states. In part A, students consider the inner products between the basis state vectors for a spin-1/2 system. In part B, students consider a state vector for an electron with complex coefficients and an overall factor N (often known as the normalization constant). They are asked to calculate the inner product of this state vector with itself. Based on our finding discussed in chapter 2, many students failed to take complex conjugate. To help students correctly calculate and interpret the inner product of this state vector with itself to the magnitude, we ask students first to relate the inner product of a *spatial* vector with itself to the magnitude. Then we ask students to consider whether the inner product of a state vector and itself can be negative or imaginary, followed by prompting students to check their calculation and resolve inconsistencies. Lastly, students are asked to generalize the difference between spatial vectors and quantum state vectors from the perspective of determining inner products.

Part C guides students to consider the need to normalize quantum state vectors. First, students are asked to determine the probability for spin up or down in the *z*-direction in terms of *N*. Then, students are asked to determine *N* by considering the total probability. Lastly, we introduce the terminology: normalization constant and then have students think about the necessity for normalization of quantum states.

In the third section, we have modified the student dialogue. The statements of the first two students were kept in order to elicit the incorrect ideas: (1) equating $|+\rangle_z$ with $|+\rangle_x$, and (2) confusing 3-dimensional laboratory space with 2-dimensional Hilbert space. The third student in the modified version argues that the inner product ${}_x\langle+|+\rangle_z$ is not zero because *"if we send an electron prepared in state* $|+\rangle_z$ *to a Stern-Gerlach apparatus with magnetic field in the x-direction, we would have 50/50 chance of measuring spin up and spin down."* The objective is to help students recognize that the linear relations between two sets of basis states are consistent with Stern-Gerlach experimental results.

3.2 Initial Development and Assessment of Tutorial *Probability Amplitude*

Originally there were two tutorials in sequence that are intended to help students understand inner products and probabilities. They are *Treating functions as vectors* and *Representations of wave functions.* A third tutorial *Probability amplitude* has been developed based on the research findings discussed in chapter 2. The primary goals of tutorial *Probability amplitude* are to help students recognize that (1) a wave function is an inner product and (2) determining probabilities for a discrete case is analogous to that for a continuous case.

When we first implemented the tutorial *Probably amplitude*, it was added as the third of the new sequence of tutorials: *Treating functions as vectors, Representations of wave functions*, and *Probability amplitude.* We then assessed the effectiveness of the new sequence of tutorials. In the subsequent academic year, the lecture instructor suggested to change the order of the tutorials in order to better synchronize with lecture instructions. Therefore, we reordered the tutorials. (The specific order will be discussed in the relevant sub-sections.) As a result, we assessed the tutorial *Probability amplitude* rather than the sequence of three tutorials.

In section 3.2.1, we describe the original sequence of tutorials *Treating functions as vectors* and *Representations of wave functions*. In section 3.2.2, we describe the first version of tutorial *Probability amplitude.* We then discuss the results from the assessment of the new sequence of three tutorials in section 3.2.3, which is followed by the results from the assessment of the tutorial *Probability amplitude.*

3.2.1 Original Tutorials on Quantum States and Inner Products

We describe the original sequence of tutorials, *Treating functions as vectors* and *Representations of wave functions*, which are intended to help students develop a functional understanding of quantum states and inner products.

3.2.1.a Tutorial *Treating Functions as Vectors*

The original tutorial *Treating functions as vectors* (see Appendix A, Figure A.3) begins with a short exercise in which students think about spatial basis vectors (in laboratory space) and dot products. Students are then given three wave functions, the ground state and the second excited state wave functions for the infinite square well, and a superposition of these wave functions. Both the algebraic expressions and graphs of the wave functions are shown. Students are asked to divide the *x*-axis into three regions, and to approximate each function as a constant through each region. They are then guided to rewrite the approximations as column vectors. They are then asked to describe strategies to improve their approximations.

The last section of the tutorial is intended to help students develop a strategy to qualitatively determine the inner product of two graphical functions. Students are first asked to approximate the value for an integral of the product of two functions and to describe their procedures. They are then told that this integral is equivalent to an inner product of two functions. Lastly, students use this method to qualitatively determine two additional inner products.

3.2.1.b Tutorial *Representations of Wave Functions*

The original tutorial *Representations of wave function* (see Appendix A, Figure A.4) starts with a case involves energy (discrete for infinite square well) basis, in which students consider different representations of the same quantum state: Dirac notation and histogram. Students are asked to determine the possible outcomes of an energy measurement and the corresponding probabilities. In the second section of the tutorial, students consider a graphical position-space wave function. They are then asked to identify the similarities and differences between the position representation and energy representation.

In the third section of the tutorial, students are guided to think about how relative phases between energy eigenstates can affect position-space probability densities. Students are given three wave functions that differ only by a relative phase. They are first asked to make predictions whether the probability densities will be the same or different. They are then shown graphs and given a chance to revisit their predictions and resolve inconsistencies.

In the last section of the tutorial, students are given a position-space wave function, both the algebraic expression and a graph. Note that the wave function is not explicitly written as a linear combination of energy eigenfunctions. Students are asked to predict the ranking of the probabilities for several energy eigenvalues. They are then guided to find a strategy to qualitatively evaluate the probabilities. This tutorial ends with a student discussion, which is intended to help students synthesize and recognize that any representation of a quantum state provides information about probabilities for all the observables.

3.2.2 Initial Development of Tutorial *Probability Amplitude*

We first discuss the initial development of the in-class worksheet of tutorial *Probability amplitude* in section 3.2.2.a. We then describe the homework of tutorial *Probability amplitude* in section 3.2.2.b.

3.2.2.a In-class worksheet

The first iteration of the tutorial *Probability amplitude* is shown in the appendix (see Appendix A, Figure A.5). It is in the context of the infinite square well potential. Students first consider inner products and probability amplitudes for energy in the first section. They are then prompted to think about these concepts for position in the second section.

The primary goals of the first section are to help students recognize that the expansion coefficients for a state vector written in the energy basis can be determined by inner products, and the modulus squares of those inner products give energy probabilities. These are guided by one of our research findings that students often recognize the probability can be found by the modulus square of the corresponding expansion coefficient, but fail to recognize the coefficient can be determined by an inner product.

At the beginning of the first section, students consider a generic quantum state written in Dirac notation. They are asked to write the state vector in terms of the energy eigenstates and to determine the corresponding coefficients. Then students are given a handout that has an expression of quantum state with numerical coefficients. Students are asked to determine the inner product between the quantum state and itself. They are also prompted to relate the energy probability and $\langle\psi_n|\varphi\rangle$, the inner product between an energy eigenstate and the state vector.

Next students are introduced the term, probability amplitude[2], an inner product between a state and an eigenstate of an observable. The term "probability amplitude" is introduced to help students relate an inner product to a probability. As discussed in chapter 2, students sometimes think that the probability is determined by simply the inner product (without a modulus square). We think that this term may support students to recognize that a modulus square is needed by drawing an analogy from classical interference, since the intensity of light is proportional to the modulus square of the amplitude of an electromagnetic wave. The quantum probability is analogous to the intensity of light, and the probability amplitude is analogous to the amplitude of an electromagnetic wave. Students then consider the probability in a case that the probability amplitude is negative.

[2] This term is introduced in McIntyre's textbook, but not in Griffiths', the textbook used in the junior-level quantum course at the UW and at many other institutions.

The second section is intended to facilitate students to recognize that determining probabilities for discrete cases is analogous to that for continuous cases, using the energy and position for the infinite square well potential as an example. The results discussed in chapter 2 suggest that many students do not differentiate between a state vector and its associated wave function. After completing the second section, students are expected to recognize that the position-space wave function is a list of coefficients of the state vector expressed in the position basis, and there the wave function can be determined by the inner product $\varphi(x) = \langle x|\varphi\rangle$.

The second section starts with a task that requires students to determine the probability of finding the particle in a specified region given a graphical position-space wave function. They are then told that the wave function can be written as an inner product $\varphi(x) = \langle x|\varphi\rangle$. The following question asks students whether or not the term probability amplitude would be appropriate to describe the wave function. They are then asked to rewrite the same position probability in terms of the inner product. Students are prompted to compare the procedure of determining probability for energy and that for position by discussing the similarities and the differences.

The last part of the second section is to help students recognize that a wave function is a list of coefficients corresponding to the position basis. Students are asked to write the state vector in terms of the position eigenstates and to determine the expansion coefficients by drawing analogy from the expressions they wrote for the energy basis in the first section. We do not expect students to use identity operators, but rather to use a semi-intuitive approach. That is, students apply the model that they have built for the discrete case to a continuous case with reasonable modification on the existing model. We hope, through exploring the formalism by themselves, that students have a relatively deep level of understanding of the analogy between discrete and continuous cases.

3.2.2.b Homework

The first iteration of the homework for *Probability amplitude* is shown in the appendix (see Appendix B, Figure B.1). It extends the material covered in the tutorial.

On the homework, students are introduced the formal formalism. Students are expected to recognize the formal formalism is consistent with what they have determined in the tutorial using a semi-intuitive approach. Similar to the tutorial, the homework question starts with energy basis. Students consider the operator $|\psi_n\rangle\langle\psi_n|$. They are asked to give an interpretation for this operator after they act that on a state vector. Students are then guided to interpret operator $\sum_n|\psi_n\rangle\langle\psi_n|$. They are expected to recognize that this operator does not change the state vector but rewrite it in the energy basis. Next, students repeat the exercise for position space. They are asked to interpret operator $\int_0^a dx|x\rangle\langle x|$.

The following exercise is intended to provide an opportunity for students to apply what they have learned from the tutorial and the first few questions on the homework. Students are asked to determine momentum probability in a specified interval given the momentum eigenstates and to write the state vector in terms of the momentum eigenstates.

The last part of the first problem requires students to generalize. They are given a hypothetical operator (for an arbitrary observable). They are asked to determine probabilities for that observable in two cases: (1) the corresponding eigenvalues are discrete, and (2) the corresponding eigenvalues are continuous.

3.2.3 Initial Assessments of the New Tutorial *Probability Amplitude*

In section 3.2.3.a, we discuss an assessment of a new sequence of three tutorials. In section 3.2.3.b, we discuss an assessment of the new tutorial alone.

3.2.3.a Initial Assessments of the New Sequence of Three Tutorials

The new sequence of three tutorials was implemented in Autumn 2015. As stated earlier, the sequence is intended to address some of the difficulties students have with quantum states and probabilities discussed in chapter 2. The new sequence of tutorials consists of *Treating functions as vectors, Representations of wave functions,* and *Probability amplitudes.* Students started to work on the new sequence in week 2 of the quarter.

We administered two tasks on exams in Autumn 2015 after students had completed the entire new sequence of tutorials. The first task was given on a midterm exam. It tested student ability to determine energy probabilities with graphical wave functions. The second task was given on the final exam. It examined student ability to apply inner product to an arbitrary observable with continuous eigenvalues. All the exams in Autumn 2015 were open-book, which was a policy decided upon the lecture instructor. Students were allowed to use both textbook and course notes.

3.2.3.a.1 Tasks with graphical wave functions

To assess the effectiveness of the new sequence of tutorials, we compare student performances from two classes: class A in Autumn 2013 and class B in Autumn 2015. Class A completed the original sequence of tutorials: *Treating functions as vectors* and *Representations of wave functions.* Class B completed the new sequence of three tutorials. Class A had Task 2.16 (see Figure 2.11 in section 2.5.3) and class B had Task 3.1, as shown in Figure 3.1. Both tasks were administered on midterm exams, but the exam in class A was closed-book and the exam in class B was open-book.

In Task 2.16, students are given three graphical wave functions. One of them describes the state of a particle in an unknown potential well, and the other two show the ground state and first-excited state. Students are asked whether the probability for this particle to be measured to have the ground state energy is greater than, less than, or equal to the probability that it is measured to have the first excited state energy.

Task 3.1 is similar to Task 2.16, but is slightly more challenging. Students are given four graphical wave functions. The first wave function represents an energy eigenstate with eigenvalue *E*. Three additional graphs are given, each represents the state of a particle in the potential well. Students were asked which of the three particles is most likely to be measured to have energy *E*.

As discussed in section 2.5.3, the energy probability can be determined by the modulus square of the inner product between the energy eigenstate and the state vector for the particle, $|\langle\psi_E|\psi\rangle|^2$. The inner products $\langle\psi_E|\psi_A\rangle$ and $\langle\psi_E|\psi_B\rangle$ are both zero. The inner product $\langle\psi_E|\psi_C\rangle$ is negative, thus particle C has the largest probability to be found to have energy *E*.

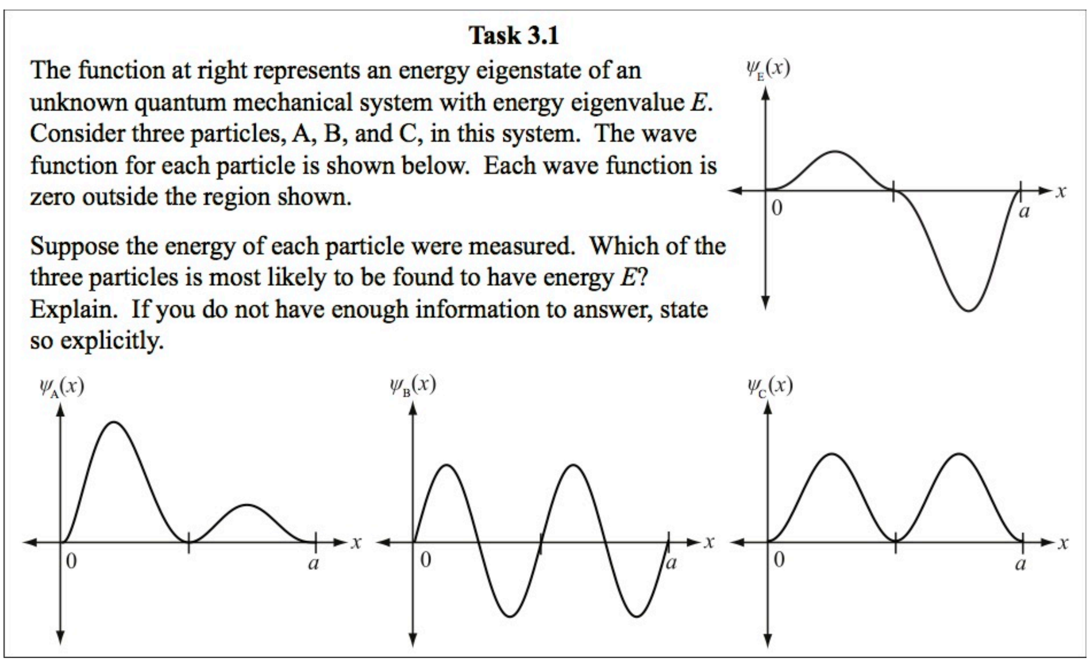


Figure 3.1. A question about energy probability given on a midterm exam in PHYS 324.

Student performance for classes A and B are shown in Table 3.16. The prevalence of students who gave a correct answer in class A is very similar to that in class B. However, the prevalence of students who gave a correct answer with correct reasoning in class B is much higher than in class A. About half (47%) of the students in class B correctly used inner products and/or symmetry to explain their answers, while only about 20% of the students in class A did so. The two-tail *p*-value from the Fisher's exact test is 0.0002, which suggests that the difference in percentages of correct answer with correct reasoning between class A and class B is extremely statistical significant (at the level of 0.001).

Table 3.16. Results from Task 2.16 and Task 3.1

	Task on exam	Tutorial instruction	Correct answer	Correct reasoning
Class A ($N = 87^{a}$)	Task 2.16 Closed-book	*Treating functions as vectors* *Representations of wave function*	53%	20%
Class B ($N = 78^{b}$)	Task 3.1 Open-book	*Treating functions as vectors* *Representations of wave function* *Probability amplitude*	58%	47%

[a]PHYS324: AUT14. [b]PHYS324: AUT15.

The results suggest that the additional tutorial had an impact on student performance. However, there are some complicated factors to the analysis since there are several differences between the classes: (1) the tutorials used in instruction, (2) the task used, and (3) the format of the exam. The task used in class B is likely somewhat more challenging than the task in class A, in part since Task 3.1 involves four wave functions rather than three. However, class B had an open-book exam while class A had a closed-book exam. Thus, it is not clear whether or not the improvement in performance is due only to the new sequence of tutorials. The results from the next task lend support to the idea that the additional tutorial was in part responsible for the difference.

3.2.3.a.2 Task with an arbitrary observable with continuous eigenvalues

Task 3.2 was used to assess student ability to determine probability for an arbitrary observable with continuous eigenvalues. The task is intended to test whether students recognize that determining probabilities involve taking inner products, and whether students recognize that an integral is needed for continuous eigenvalues. We did not give a task that is comparable enough to Task 3.2 so that we do not intend to use statistical tests for comparison. Instead, we qualitatively compare results from similar tasks. Besides, we identify student difficulties that persist even after tutorial instruction.

In Task 3.2, students consider an arbitrary operator $\hat{Q}$. The eigenvectors and eigenvalues of operator $\hat{Q}$, and the state vector of a particle are given. Students are asked to describe how to find the probabilities for operator $\hat{Q}$.

Task 3.2

Consider a Hermitian operator $\hat{Q}$ with a continuous non-degenerate spectrum of eigenvalues. Let the eigenstates and eigenvalues of $\hat{Q}$ be given by $|\psi_q\rangle$ and q, respectively. Suppose the state of a quantum system is described by $|\varphi\rangle$. Describe how to find the probabilities of measuring the observable corresponding to the operator $\hat{Q}$. Explain.

Figure 3.2. A task requires determining probability for a continuous case

As discussed in chapter 2, to find the probabilities, one needs to take the inner product $\langle\psi_q|\varphi\rangle$, and then integrate the modulus square of the inner product over a region $\int_{q_1}^{q_2}|\langle\psi_q|\varphi\rangle|^2 dq$ since the eigenvalues of operator $\hat{Q}$ are continuous.

Table 3.17 shows student performance on Task 3.2. Only about 5% of the students gave a completely correct answer. About half (54%) of the students recognized that the inner product $\langle\psi_q|\varphi\rangle$ is needed. However, many (about 42% of all students) did not write an integral, and some (about 7% of all students) wrote incorrect bounds or did not include bounds for the integral. Another common incorrect answer (17% of all students) was that the expression for probability involves acting the operator on the state vector, such as $|\langle\psi_q|\hat{Q}|\varphi\rangle|^2$.

Table 3.17. Student performance on Task 3.2

		$N = 76$[a]
Completely correct		5%
Common incorrect answers	Wrote the inner product and integral, but incorrect (or no) bounds	7%
	Wrote the inner product without integral	42%
	Used the operator $\hat{Q}$	17%

[a]PHYS324: AUT15.

As discussed in sections 2.3 and 2.4, many students do not always recognize that determining probabilities involving taking inner products, especially for the case of continuous eigenvalues. Even in a discrete case when students were asked to determine the probability for angular momentum (see Task 2.8 in section 2.3), only about one-quarter of the students (explicitly or implicitly) answered that an inner product is needed. In contrast, about half of the students wrote an inner product on Task 3.2. This suggests that the new sequence of tutorials may be effective at helping students recognize that determining probabilities involves taking inner products.

Although many students took an inner product, only 12% of the students wrote an integral for operator $\hat{Q}$ with continuous eigenvalues. It is possible that some students missed the word "continuous" in Task 3.2. It is also possible that many students still have difficulty with determining probabilities for continuous cases after both lecture and tutorial instruction.

3.2.3.b Initial Assessment of the New Tutorial *Probability Amplitude*

The previous section presented results from example tasks that tested the impact of the entire three-tutorial sequence. In this section, the assessment focuses on the tutorial *Probability amplitude* alone.

In Autumn 2016, a couple changes were made to the tutorial section of the course. The lecture instructor suggested changing the sequence of tutorials so that the tutorial *Treating functions as vectors* was after tutorials *Representations of wave functions*, *Probability amplitude*, and a tutorial on time dependence. The purpose of this change was to have the tutorials better synchronized with the lecture materials. Another change implemented by the lecture instructor was that there be no questions directly related to tutorials on exams. All questions on exams were based on the material covered in lectures.

To assess the effectiveness of tutorial *Probability amplitude,* we administered tasks on online surveys both before and after students had completed the tutorial. The tasks (Task 2.9, Task 2.10, and Task 2.11) given to students before they completed the new tutorial have been discussed in section 2.4. All three tasks require students to translate expressions between Dirac and wave function notations. After students completed the tutorial, they were given variants of Task 2.9 and Task 2.10, as well as Task 2.11.

We first discuss the results from the variant of Task 2.10 (in part 1 below), followed by the results of Task 2.11 (in part 2). Lastly, we report results from the variant of Task 2.9 (in part 3).

3.2.3.b.1 Comparison of student performance on Task 2.10 before and after tutorial instruction

Task 3.3, the variant of Task 2.10 (see Figure 2.7 in section 2.4), is shown in Figure 3.3. Task 2.10 is also shown for comparison. These two tasks probe student ability to translation inner products from Dirac notation to wave function notation. In Task 2.10, students consider three equations: two for inner product $\langle x|\psi\rangle$ and one for $\langle\varphi_n|\psi\rangle$. Answer (c), the equation for inner product $\langle\varphi_n|\psi\rangle$ is correct. Answer (a) is used to determine whether students confuse the position operator with the position eigenstate. Answer (b) is correct. To recognize that answer (b) is correct, students need to either recognize that the position eigenfunction in position space in a delta function, or recognize that the inner product $\langle x|\psi\rangle$ is equal to $\psi(x)$.

Task 2.10

Indicate all of the following equations that are correct.
(a) $\langle x|\psi\rangle = \int x\psi(x)dx$
(b) $\langle x|\psi\rangle = \int \delta(x-x')\psi(x')dx'$
(c) $\langle\varphi_n|\psi\rangle = \int \varphi_n^*(x)\psi(x)dx$

Task 3.3

Indicate all of the following equations that are correct.
(a) $\langle p|\psi\rangle = \int p\psi(x)dx$
(b) $\langle p|\psi\rangle = \int \exp(-ipx)\psi(x)dx$
(c) $\langle\varphi_n|\psi\rangle = \int \varphi_n^*(x)\psi(x)dx$

Figure 3.3. Tasks about translating inner products between notations

Task 3.3 asks for the correct expression(s) for the inner product $\langle p|\psi\rangle$. Analogous to Task 2.9, answer (a) in Task 3.3. is intended for students who confuse the eigenstate of an operator with the operator, and answer (b) is correct since the position-space momentum eigenfunction is $\exp(ipx)$. Answer (c) in Task 3.3 is identical to answer (c) in Task 2.10.

In order to test whether the tutorial is effective at addressing the difficulty with differentiating between the eigenstate of an operator and the operator itself, we assess student ability to identify the incorrect answers in both tasks. Moreover, it has been our experience that students tend to be more familiar with position operator and its eigenfunction than with the momentum operator and the corresponding eigenfunction. Since the tutorial is not intended to directly help students recognize that the position-space momentum eigenfunction is exp (ipx), we do not compare the prevalence of correct answer before and after tutorial instruction. Both tasks were given to the students in the same class, thus we can compare the number of students who chose the incorrect answer on the pretest but did not choose it on the post-test and the number of students who did not choose the incorrect answer on the pretest but chose it on post-test. Students who did not answer both tasks were not counted.

Table 3.18. Number of students who did or did not include answer (a) on variants of Task 2.10

$N = 62$[a]		Post-test	
		Included answer (a)	Did not include
Pretest	Included answer (a)	10	21
	Did not include	8	23

[a]PHYS 324: AUT16

As shown in Table 3.18, 21 students incorrectly chose answer (a) on the pre-test but did not choose it on the post-test, while 8 students did the opposite. Others did not change their answers from pretest to post-test. To assess whether or not the difference in numbers of students who changed their answers is simply due to random fluctuation, we used the McNemar's test. The two-tailed p-value is 0.0259, which suggests that the difference is statistically significant at the level of 0.05. The results suggest that the tutorial seems effective at addressing the difficulty with differentiating between the eigenstate of an operator with the operator itself.

Caveats on results: Note that the post-test involves momentum eigenstates rather than position eigenstate. Momentum operator and its eigenstates are not discussed as extensively as position operator. We believe that the post-test may be somewhat harder for students and thus the results represent a gain in performance. We plan to test this by reversing the pretest and post-test in the future.

3.2.3.b.2 Comparison of student performance on Task 2.11 before and after tutorial instruction

Task 2.11 (see Figure 2.7 in section 2.4) was given to students as both pre-test and post-test. Both of the answers are correct. Answer (a) involves an integral over the momentum space $|\psi\rangle = \int |p\rangle\langle p|\psi\rangle dp$, and answer (b) involves an integral over the position space $|\psi\rangle = \int \psi(x)|x\rangle dx$. We first compare the number of students who correctly included answer (a) on the pretest but did not include it on the post-test and the number of students who did the opposite. We then compare the number of students who correctly included answer (b) on the pretest but did not include it on the post-test and the number of students who did the opposite.

Table 3.19. Number of students who did or did not include answer (a) on Task 2.11

$N = 60$[a]		Post-test	
		Included answer (a)	Did not include
Pre-test	Included answer (a)	21	7
	Did not include	19	13

[a]PHYS 324: AUT16

Table 3.19 shows the numbers of students who did or did not include answer (a) on pre-/post-test. The two-tailed *p*-value from the McNemar's test is 0.0310, which suggests that the difference in numbers of students who switched answers is statistical significant at the level of 0.05.

Table 3.20. Number of students who did or did not include answer (b) on Task 2.11

$N = 60^a$		Post-test	
		Included answer (b)	Did not include
Pre-test	Included answer (b)	8	6
	Did not include	21	25

[a]PHYS 324: AUT16

Table 3.20 shows the number of students who did or did not include answer (a) on pre-/post-test. The two-tailed *p*-value from the McNemar's test is 0.0071, which suggests that the difference in numbers of students who switched answers is statistical significant at the level of 0.01.

The results suggest that the tutorial *Probability amplitude* is effective at helping students recognize the correct expressions for state vectors written in the momentum basis and in position basis and at helping students recognize that the wave function is an inner product.

3.2.3.b.3 Comparison of student performance on Task 2.9 before and after tutorial instruction

On the post-test, a variant of Task 2.9 was given. The original Task 2.9 asks students to choose from a list of expressions the ones that represent probability density. The variant of Task 2.9, used as a post-test, asks for expressions for the probability of finding the particle in the left half of the well. Each answer choice in the variant is analogous to the corresponding answer choice in the original task (see Figure 2.7 in section 2.4).

On both tasks, answers B and D are correct, and answers A and C are incorrect. Answer A tests to determine whether students confuse the inner product $\langle\psi|\psi\rangle$ with the probability density

$|\psi(x)|^2$. Answer C is intended to determine whether students confuse the inner product $\langle\varphi_1|\varphi_2\rangle$ with the product $\varphi_1(x)\varphi_2(x)$.

We first compare the number of students who correctly included answer (a) on the pretest but did not include it on the post-test and the number of students who did the opposite. We then do the same comparison for answer (c).

Table 3.21. Number of students who did or did not include answer (a) on variants of Task 2.9

$N = 61$[a]		Post-test	
		Included answer (a)	Did not include
Pre-test	Included answer (a)	25	12
	Did not include	17	7

[a]PHYS 324: AUT16

Table 3.21 shows the numbers of students who did or did not include answer (a) on pre-/post-test. There are slightly more students who did not include the incorrect answer (a) on pretest but included it on the post-test than students who did the opposite. However, the two-tailed *p*-value from the McNemar's test is 0.4576, which suggests that the difference in numbers of students who switched answers is not statistical significant.

Table 3.22. Number of students who did or did not include answer (c) on variants of Task 2.9

$N = 61$[a]		Post-test	
		Included answer (c)	Did not include
Pre-test	Included answer (c)	3	18
	Did not include	13	27

[a]PHYS 324: AUT16

Similarly, there are slightly more students who did not include the incorrect answer (c) on pretest but included it on the post-test than students who did the opposite. However, the difference in numbers of students who switched answers is not statistical significant since the two-tailed *p*-value from the McNemar's test is 0.4725.

The results suggest that the confusion between the inner product $\langle\psi|\psi\rangle$ with the probability density $|\psi(x)|^2$ and the confusion between the inner product $\langle\varphi_1|\varphi_2\rangle$ and the product $\varphi_1(x)\varphi_2(x)$ persist even after tutorial instruction. It seems that many students still do not differentiate between a state vector $|\psi\rangle$ and its associated wave function $\psi(x)$.

3.2.4 Summary and Implications of the Initial Assessments

The results from the initial assessments seem to suggest that the new tutorial *Probability amplitude* and the new sequence of tutorials are effective at helping students recognize that (1) determining probabilities involving taking inner products, (2) the correct expressions for state vectors written in the momentum basis and in position basis and (3) the wave function is an inner product.

However, some issues persist even after the intervention. Many students still seem to have difficulty with determining probabilities for continuous cases. Many do not differentiate between the inner product $\langle\psi|\psi\rangle$ and probability density $\psi^*(x)\psi(x)$.

3.3 Modifications and Additional Assessments

It has been our experience that students do not construct a concept or develop a reasoning skill at once, but rather in stages. It is necessary for students to revisit the same concepts at gradually increasing levels of challenge. Therefore, we modified tutorials *Representations of wave functions* and *Probability amplitude.* In this section, we discuss how the results discussed in section 3.2. guide the modifications of these tutorials. We also report additional results of assessment of modified tutorials.

3.3.1 Modified Tutorial *Representations of Wave Functions*

The results from the initial assessment of the new sequence of tutorials suggest that many students still have difficulty with determining probabilities with continuous cases. Therefore, the modifications to the tutorial *Representations of wave functions* are intended to guide students to think about the analogy between discrete and continuous cases before they work through *Probability amplitude*. Specifically, the modified tutorial prompts students to compare a finite sum for an energy probability and a Riemann sum for a position probability.

In the first section, after students consider the histogram of a state vector written in the energy basis, they are asked to sketch another histogram for energy probabilities. Students are then asked to describe their methods for finding the probability that energy is smaller than a particular value. They are expected to arrive at a sum of the probability for each individual energy.

In the second section, students are asked to sketch the probability density given a graphical position-space wave function. Using the probability density graph, students need to identify the probability of finding the particle in each of the three small regions from the probability density graph. Students are then asked to determine the probability of finding the particle in a large region that contains the three small regions. They are then prompted to compare the probability of measuring the energy a range and the probability of measuring the position in a range. In addition, students consider the similarities and differences between the coefficients for energy basis and the wave function.

3.3.2 Modified Tutorial *Probability Amplitude*

We modified both the tutorial and the homework of *Probability amplitude.*

In the modified tutorial (see Appendix A, Figure A.9), a section of spatial vectors was added at the beginning. Students are guided to relate the components of a spatial vector to dot products, and to interpret dot products as projections. In the second section, students are prompted to think about how the coefficients for a quantum state written in the energy basis are analogous to the projection of two spatial vectors. In the third section, students are guided to think about the similarity between the coefficients for the energy basis and the position-space wave function. The objective is to help to understand probability amplitudes (determined by an inner product) by drawing an analogy from spatial vectors.

Similar to the changes made to the tutorial *Representations of wave functions* (see section 3.3.1), students are again guided to think about the similarities between determining probability for discrete cases and that for continuous cases. In the second section, students are asked to write an expression in Dirac notation for the energy probability in a range of energy eigenvalues rather than for the energy probability for a particular value. Students are then, in the third section, asked to compare the expression for the probability of measuring the position in a range to what they wrote in the second section for energy.

At the end of the tutorial, students are explicitly prompted to discuss the difference between a state vector and its associated wave function. The change was motivated by the results from the initial assessment discussed in section 3.2 that many students still confuse the inner product $\langle\psi|\psi\rangle$ with $\psi^*(x)\psi(x)$, which suggest that they do not differentiate between $|\psi\rangle$ and $\psi(x)$.

We have also added a statement about the interpretation of the inner product $\langle x|\psi\rangle$ toward the end of the tutorial. This statement is intended to help students interpret the wave function as a projection by drawing the comparison from the inner product between the energy eigenstate and the state vector. Finally, students are prompted to generalize the procedures for determining probabilities for an arbitrary observable in two conditions, one with discrete eigenvalues and another with continuous eigenvalues.

In the homework (see Appendix B, Figure B.4), a new question was added at the beginning. Students are given several inner products: $\langle\psi_n|\varphi\rangle$, $\langle\varphi|\varphi\rangle$, $\langle\psi_n|\psi_m\rangle$, $\langle x|\psi\rangle$, and $\langle x|x'\rangle$. They are asked to write expressions for the results of those inner products. At the end of the second question, students are guided to use the identity operator in position basis to rewrite some of the inner products given in the first question. Then are asked to compare their answers in both questions and resolve any inconsistencies. This exercise is intended to address the confusion between inner product $\langle\varphi|\varphi\rangle$ and $\psi^*(x)\psi(x)$, and the confusion between $\langle\psi_n|\psi_m\rangle$ and $\psi_n^*(x)\psi_m(x)$.

We have also modified the exercise for the momentum probability. We made this changed since this exercise was graded and we found many students did not do well. We switched the order of the two questions in this exercise. In the modified version of homework, students are to first write the state vector in terms of the momentum eigenstates, and then to determine momentum probability in a specified interval. We believe this change may help students interpret the inner product as the coefficients in the momentum basis.

3.3.3 Additional Assessments

We discuss additional assessment after modifications on tutorials *Representations of wave functions* and *Probability amplitude*.

Task 2.12 (see Figure 2.9 in section 2.4) was used to assess the effectiveness of the modified tutorials. On this task, students consider a position-space wave function for a particle in the infinite square well. They are asked to determine the ground state energy probability in part A and the position probability in a specified region in part B.

The same task was given on exams in three different classes. Students in class A did not have tutorials that are intended to help students with quantum states and probabilities, but had completed a sequence of tutorials on time dependence. Students in class B completed a sequence of three tutorials *Treating functions as vectors, Representations of wave functions* and *Probability amplitude*. Students in class C completed tutorials *Representations of wave functions* and *Probability amplitude*. The tutorial *Probability amplitude* was given after students had completed *Representations of wave functions* and the entire sequence of tutorials on time dependence. This change was made in order to have tutorial materials synchronize with lecture materials better.

Another major difference among these classes is that the grade proportion for tutorial questions on exam. In both classes A and B, the tutorial questions count about one-third of the total grades of the exams. In contrast, the tutorial questions count only about 7% of the total grades of the exams in class C.

Table 3.23. Student performances on Task 2.12 in different classes

	Tutorial instruction	Grade proportion	Correct energy probability	Correct position probability
Class A ($N = 71^a$)	A sequence on time dependence	30%	55%*	58%
Class B ($N = 25^b$)	*Treating functions as vectors* *Representations of wave function* *Probability amplitude*	33%	68%**	84%
Class C ($N = 98^c$)	*Representations of wave function* A sequence on time dependence *Probability amplitude*	7%	46%	74%

[a]PHYS 324: AUT13. [b]PHYS 324: SUM17. [c]PHYS 324: AUT17.

*3% no mod square. **12% no mod square.

Student performances in classes A, B, and C are shown in Table 3.23. Comparing class A and class B, both the percentage of correct energy probability and the percentage of correct position probability are greater for class B. However, only the difference in the percentages of correct position probability is statistical significant (the two-tail *p*-value from Fisher's exact test for correct energy probability is 0.3463 and for correct position probability is 0.0274).

Comparing class A and class C, the percentage of correct position probability is greater for class C, but the percentage of correct energy probability is smaller. The two-tail *p*-value from Fisher's exact test for the correct energy probability is 0.2272 and for the correct position probability is 0.0302. This suggest that the difference in the percentages of correct position probability is statistical significant, but the difference in percentages of correct energy probability is not statistical significant.

The results suggest that the sequence of new tutorials, or the two tutorials are effective at improving student ability to determine position probability given a position-space wave function. However, the difficulty with determining energy probability seem to persist even after lecture and tutorial instructions. We suspect that the significant difference in grade proportions of tutorial-based questions on exams might have also impacted the results to some extent. Since the tutorial questions only counted about 7% of the total grade on an exam in class C, it is likely that students in class C did not make enough efforts due to a lack of incentives.

3.4 Summary

In this chapter, we described the iteration of tutorial development and discussed preliminary results from assessment of tutorials on quantum states and inner products. The development of tutorials was driven by the results from investigating student understanding of the use of inner products to determine probabilities, discussed in chapter 2.

The results from chapter 2 suggest that many students do not recognize that determining probabilities for discrete and continuous cases are analogous. This seems due to a lack of functional understanding of quantum states and inner products in vector space. Therefore, we developed tutorials at both sophomore- and junior-level. At the sophomore-level, we modified tutorial *Dirac notation*. At the junior-level, we developed a new tutorial *Probability amplitude* and modified the existing tutorials on quantum states and inner products.

Our preliminary results from the assessment suggest that the new sequence of tutorials seem to have some positive impact on student ability to translate expressions from Dirac notation to wave function notation, and to determine position probability. However, it seems that many students still have difficulty recognizing that determining probabilities involve taking inner products after lecture and tutorial instruction. It is also worthwhile to mention that there were some complications for the implementation of the new versions of tutorials, such as a different order of tutorials and smaller incentives for completing tutorials. These factors may have influenced the results. Additional assessments and further investigation are needed.

Chapter 4. Student Understanding of Relative Phase in Superposition States

[Note: This chapter is based on Ref. [18], which was written in close collaboration with Paul J. Emigh and Peter S. Shaffer. It expands on the details of the study and includes a follow-up investigation that is not included in the published paper.]

A quantum state can be expressed as a linear combination of basis states. The coefficients, also known as probability amplitudes, are complex numbers. The modulus squared of a probability amplitude gives the probability of measuring the observable corresponding to the basis state. Typically, the probability amplitudes are represented by positive real numbers multiplied by complex exponentials. The argument of a complex exponential is called the phase. A state can have an overall phase, but the overall phase does not affect any probabilities since that phase always cancels when determining probabilities (by calculating the modulus squared). However, the relative phases between the basis states can have measurable effects. For example, the two spin states $\frac{1}{\sqrt{2}}|+\rangle_z + \frac{1}{\sqrt{2}}|-\rangle_z$ and $\frac{1}{\sqrt{2}}|+\rangle_z + e^{i\pi}\frac{1}{\sqrt{2}}|-\rangle_z$, which differs only in the relative phases between the basis states, have the same probability of measuring the *z*-component of spin but different probabilities of measuring the *x*-component.

The example above illustrates how the basis states $|+\rangle_z$ and $|-\rangle_z$ can "interfere" differently with each other depending on the relative phase between them. It thus demonstrates the important role that complex numbers play in quantum mechanics.

The concept of relative phase has relevance in multiple ways. For example, it underlies an important but challenging idea in quantum mechanics, time dependence. When a superposition of energy eigenstates evolves in time, the relative phase also changes in time. This results in time-dependent probabilities. For example, the relative phase in the time-dependent spin state described by $\frac{1}{\sqrt{2}}e^{-i\omega t}|+\rangle_z + \frac{1}{\sqrt{2}}e^{i\omega t}|-\rangle_z$ changes over time. Hence, the probability of measuring the *x*-component (or some other component) of spin depends on time for this state. A functional understanding of relative phase is thus a key step to grasping time dependence in quantum mechanics.

Textbooks on quantum mechanics that use a spins-first approach (including McIntyre's textbook used in the PHYS 225 course at the UW) often start with Stern-Gerlach (SG) experiments of spin-1/2 systems. The idea behind a spins-first approach is that the mathematical calculations for spin-1/2 systems are relatively straightforward, thus this approach allows students to focus on fundamental principles of quantum mechanics. Students are guided to learn the concepts and principles in the context of concrete experiments. In McIntyre's textbook, relative phase is introduced in the context of SG experiments. The textbook demonstrates how to determine the relative phases based on the experimental results. Thus, students are expected to be able to relate relative phase to experimental results.

This chapter describes our investigation into student ability to relate quantum formalism to real-world phenomena in the context of superposition states. In particular, we examine the extent to which students are able to make a connection between relative phase and experimental results. We vary the contexts and prompts of the tasks in order to identify potential underlying difficulties students have with reasoning about relative phase in superposition states.

4.1 Prior Research

There has been a fair amount of research that focuses on investigating and improving student understanding of time dependence [6-9,26], which is a key idea in quantum mechanics that builds on superposition and relative phase. Much of this body of research examines student ability to qualitatively predict the time dependence of wave functions and the time dependence of probabilities for different observables. Some of the findings from prior research in this area have shown that many students have difficulty in interpreting the role of complex numbers and phases in the context of time dependence. For example, some students seem to believe that only the real component of a wave function contribute to determining probabilities and the imaginary component can be ignored [6]. Some students assign a single time-dependent phase factor for a superposition of energy eigenstates [2,6]. These results suggest that difficulty with complex numbers and time-dependent phase factors may impede student reasoning with time dependence.

Although superposition and relative phase underlie the concept of time dependence, research that specifically focuses on this aspect (without time dependence) is less extensive. Some research has showed that many students have difficulty in distinguishing between a pure ensemble (of particles in a superposition state) and a mixed ensemble [11]. Many students do not recognize that a pure ensemble and a mixed ensemble are experimentally distinguishable. Recently, there is an increasing effort on using interactive simulations to improve student understanding of quantum interference phenomena in the contexts of double-slit interference and single-photon interference [12-14]. However, these studies tend to focus on student reasoning about the interference patterns under different experimental conditions, rather than the mathematical formalism behind the phenomena.

We have only found two studies directly investigating student ability to relate mathematical formalism to physical phenomena in the context of superposition and relative phase. M. Michelini and G. Zuccarini have shown that when students are asked to find the coefficients for spin states based on experimental results, most use real numbers and neglect phase [15]. Close *et al.* have found that nested phasor diagrams can be used productively to represent two complex coefficients for spin-1/2 states in order to help students visualize the relative phase between two basis states [16].

There are some studies on student fluency with complex numbers. Smith *et al.* for example, investigated student fluency determining square and modulus square of complex numbers in both rectangular form $x + iy$ and exponential form $re^{i\theta}$ [68]. They identified common errors students made in the computations. They also found that many students have difficulty with changing the forms of a complex number (e.g., change from rectangular to exponential form).

In summary, previous studies on student understanding of quantum mechanics tend to center either on conceptual understanding or on mathematical sophistication. There is little research that specifically focuses on student ability to relate the formalism to real-world quantum phenomena. In particular, little research on student ability to relate relative phase and experimental results has been done.

4.2 Overview and Goals

In this chapter, we describe a study into student understanding of relative phase in superposition states. In particular, we probe the extent to which students are able to make a connection between relative phases and experimental results.

In section 4.3, we discuss student understanding of mathematical behavior of a complex exponential. In particular, we examine student ability to reason about the trajectory of a complex exponential function in the complex plane as the argument increases.

Sections 4.4 and 4.5 together discuss an investigation into student understanding of the measurable effects of relative phase. In section 4.4, we discuss results from tasks that involve open-ended questions. The tasks were used to examine whether or not students recognize that superposition states differ only by a relative phase can be experimentally distinguishable. We chose discrete (spin) and continuous (position) cases to test whether or not student reasoning is context-dependent. The results from the open-ended questions motivated the research that probed student ability to compare probabilities or modulus squared, discussed in section 4.5. Students were explicitly prompted to compare probabilities or modulus squared in three different contexts. The first two involve quantum mechanics: spin and infinite square well. The third is in a purely mathematical context. These three contexts provide insights into how student reasoning about mathematical expressions in the context of superposition is similar to and/or different from that in the purely mathematics context.

The findings from the tasks that probe student reasoning in quantum and mathematical contexts suggest that student views about the role of relative phases may impact student responses. In section 4.6, we discuss a follow-up investigation into student views about the roles of complex numbers in quantum mechanics and electrodynamics.

Lastly, we summarize our research findings in section 4.7.

4.3 Student Understanding of the Mathematical Behavior of Complex Exponentials

This section presents results from a task used to examine student ability to reason about complex exponentials in the complex plane. Task 4.1, as shown in Figure 4.1, was administered as part of an online survey in PHYS 225 after lecture instruction on time dependence.

In Task 4.1, students consider a complex exponential function $f(s) = e^{-is}$, where the variable s is real. Students are shown five answer options and are asked to choose the one that best describes the behavior of the function as the variable s increases. There are five answer options. Each answer option describes how the magnitude of the function changes (e.g., increases, decreases, oscillates, stays the same) as s increases. The last two answer options also describe the direction (clockwise or counter-clockwise) that the function rotates in the complex plane.

To answer, students need to know that the magnitude of this function can be determined by $\sqrt{e^{is}e^{-is}} = 1$. Thus, it stays the same as s increases. Due to the negative sign in the exponent, $f(s)$ rotates clockwise when s increases. One can also check this using Euler's formula $e^{-is} = \cos(s) - i\sin(s)$. When s is zero, $f(0) = 1$. When s increases from zero, the real part decreases while the imaginary part becomes negative. Thus, $f(s)$ initially moves to the fourth quadrant from the positive real axis. Option D is correct.

Task 4.1

Which of the following statements best describes the behavior of the function $f(s) = e^{-is}$ as the variable s increases? (Assume s is real.)

A. The magnitude of $f(s)$ strictly increases.
B. The magnitude of $f(s)$ strictly decreases.
C. The magnitude of $f(s)$ oscillates back and forth.
D. $f(s)$ rotates clockwise in the complex plane, with the same magnitude.
E. $f(s)$ rotates counter-clockwise in the complex plane, with the same magnitude.

Figure 4.1. A task involving a complex exponential given in PHYS 225.

The percentage of students giving each answer is shown in Table 4.24. The correct answer is bolded. About one-third (32%) of the students correctly chose the answer that the function $f(s)$ rotates clockwise with the same magnitude. About 27% correctly answered that the magnitude does not change but incorrectly stated the function rotates counter-clockwise. Some of these students explained that "it [the function] rotates in the opposite direction due to the negative sign." They seemed to use a memorized rule without knowing how to apply it. Others did not explicitly explain why they thought the function rotates counter-clockwise.

The remaining students (who did not choose either D or E) seemed to fail to recognize that the magnitude of the function is constant. About 23% chose the answer that the magnitude oscillates. Many of these students explained that since the complex exponential can be written as $\cos s - i\sin s$, the magnitude oscillates. It is likely that these students did not recognize the magnitude of $\cos s - i\sin s$ is one, some may have misinterpreted the statement to ask about whether the magnitudes of the real and imaginary parts oscillate. About one-fifth of the students chose the answer that the magnitude strictly increases or that the magnitude strictly decreases. These students seemed to confuse the complex exponential with an exponential (e.g., e^{-s}), which changes monotonically as s increases.

Table 4.24. Percentage of each answer on Task 4.1

	$N = 226$[a]
A (magnitude increases)	4%
B (magnitude decreases)	14%
C (magnitude oscillates)	23%
D (same magnitude, clockwise)	**32%**
E (same magnitude, counter-clockwise)	27%

[a]PHYS 225: WIN17 and WIN18.

Summary: The results above demonstrate the extent to which students understand how a complex exponential function changes in a complex plane when the argument increases. Many students do not seem to recognize that the magnitude of a complex exponential function stays constant and that it can be represented as rotating in the complex plane. Some confuse a complex exponential function with an exponential function. It is also common that many students incorrectly relate the sign of the argument to the direction of rotation.

4.4 Student Ability to Recognize the Measurable Effects of Relative Phase

In this section, we discuss results from two tasks that examine student ability to relate relative phase to experimental results. The questions are open-ended. In each question, students are given two quantum states that differ only by a relative phase. They are asked whether or not those states are experimentally distinguishable. Each question was given as part of an online survey in both sophomore- and junior-level courses after lecture instruction on relevant topics over several academic quarters. In one academic quarter, one of the tasks was given on a midterm exam in PHYS 225.

4.4.1 Task Design

PHYS 225 used McIntyre's textbook in which various Stern-Gerlach experiments (including measurements on different components of spin) are introduced. In particular, the course lectures and textbook explicitly discussed how to use the experimental results to infer the coefficients for the expressions of the S_x (and S_y) basis states written in the S_z basis. Thus, the students had seen situations in which the relative phases affect spin probabilities. Students then study position-space wave functions at the end of PHYS 225. [Experiments associated with a particle in a position-space potential are not explicitly discussed in lecture or textbook.] Given the course materials in PHYS 225, we hypothesized that the student responses are context dependent. To test this hypothesis, we used spin and the infinite square well potential contexts.

Prior research has shown some differences in student performances on computations of complex numbers with rectangular and exponential forms [68]. Therefore, we used both forms in each context to test whether or not the results depend on how the complex coefficients are expressed.

Task 4.2

Version A: Consider two ensembles of spin-1/2 particles:
The particles in ensemble A are all described by the state vector $|\psi_A\rangle = \frac{1}{2}|+\rangle_z + \frac{\sqrt{3}}{2}|-\rangle_z$. The particles in ensemble B are all described by the state vector $|\psi_B\rangle = \frac{1}{2}|+\rangle_z + i\frac{\sqrt{3}}{2}|-\rangle_z$.

Is there an experiment that can be used to distinguish the particles in ensemble A from the particles in ensemble B? If your answer is yes, describe the experiment. If not, explain why not.

Version B:
Identical to version A except that the complex coefficients are written in the exponential form:
$|\psi_A\rangle = \frac{1}{2}|+\rangle_z + \frac{\sqrt{3}}{2}|-\rangle_z$ and $|\psi_B\rangle = \frac{1}{2}|+\rangle_z + e^{i\pi/2}\frac{\sqrt{3}}{2}|-\rangle_z$.

Task 4.3

Version A: Consider two ensembles of identical particles in infinite square well potentials. There is one particle in each well.
The particles in ensemble C are all described by the wave function $\psi_C(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$.
The particles in ensemble D are all described by the wave function $\psi_D(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + i\frac{1}{\sqrt{2}}\varphi_2(x)$.

Is there an experiment that can be used to distinguish the particles in ensemble C from the particles in ensemble D? If your answer is yes, describe the experiment. If not, explain why not.

Version B:
Identical to version A except that the complex coefficients are written in the exponential form:
$\psi_C(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$ and $\psi_D(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + e^{i\pi/2}\frac{1}{\sqrt{2}}\varphi_2(x)$.

Figure 4.2. Tasks of open-ended questions about relative phase

Figure 4.2 shows the two tasks, Task 4.2 and Task 4.3. Each consists of an open-ended question about the impact of relative phase on real-world measurements of spin or position. In Task 4.2, students consider two ensembles of spin-1/2 particles. The particles in each ensemble are represented by a superposition state written in the S_z basis. The coefficients have identical magnitudes but different phases. In version A, the complex coefficient is written in the rectangular form. In version B, it is written in the exponential form. In each academic quarter, we varied the coefficients to assess whether any difficulties identified occur only in a particular case. For example, in some cases the coefficients for $|-\rangle_z$ are purely real but negative rather than purely imaginary.

In order to answer whether the states can be distinguished, students need to recognize that although the probabilities of measuring either $+\hbar/2$ or $-\hbar/2$ for S_z are identical for the two spin states (since the coefficients for both states have identical magnitudes), the result for measuring spin can be different in the x or y direction. The probability of measuring $+\hbar/2$ for S_x, for example, can be found by taking the modulus square of the inner product between the eigenstate $|+\rangle_x$ and the state vector $|\psi\rangle$: $\left|\,{}_x\langle+|\psi\rangle\right|^2$. Since the given states are written in the S_z basis, students need to know how to convert from the S_x basis to the S_z basis. Hence, the probabilities of measuring either $+\hbar/2$ or $-\hbar/2$ for S_x are different since the inner products yield values with different magnitudes. In general, the probabilities of measuring either $+\hbar/2$ or $-\hbar/2$ along an arbitrary direction can also be different. Thus, one can distinguish the particles from ensembles A and B by measuring S_x, or the spin component along a random direction.

Task 4.3 is analogous to Task 4.2, but in the context of position space. In Task 4.3, students consider position-space wave functions for particles in the infinite square wells. Similar to Task 4.2, they are asked whether or not the particles in ensembles C and D are experimentally distinguishable. In this case, the probability of measuring the ground state energy and the first excited state energy are $1/2$ for the particles in each ensemble. However, the particles in ensembles C and D have different position-space probability densities because of the difference in relative phases of the wave functions. By convention, the energy eigenfunctions for the infinite square well are taken to be real, so the probability density for particles in ensemble C can be written as $1/2\,(\varphi_1^2(x) + \varphi_2^2(x) + 2\varphi_1(x)\varphi_2(x))$ and the probability density for particles in ensemble D as $1/2\left(\varphi_1^2(x) + \varphi_2^2(x)\right)$. Hence, the probability density for the particles in ensemble D is symmetric about the midpoint of the well, whereas for ensemble C it is asymmetric, with higher probability on the left. Thus, one can distinguish the particles from those two ensembles by measuring position.

4.4.2 Overall Performance

The results on both tasks do not seem to vary substantially from one academic quarter to another even though in some versions the coefficients are purely imaginary and in others they are purely (negative) real. There is also no significant difference between the performance of sophomore- and junior-level students. Therefore, results from different courses and academic quarters are aggregated, as shown in Table 4.25.

Table 4.25. Student performance on Task 4.2 and Task 4.3

Administration	Context	Form of complex number	Population	Correct answer	Correct answer and reasoning
Online survey	Spin	Rectangular	$N = 201$[a]	60%	36%
		Exponential	$N = 40$[b]	55%	38%
	Infinite square well	Rectangular	$N = 150$[c]	47%	18%
		Exponential	$N = 281$[d]	56%	16%
Exam	Spin	Exponential	$N = 189$[e]	61%	41%

[a]PHYS 225: Win17, PHYS 324: Sum16, and PHYS 325: Win17. [b]PHYS 225: Sum17
[c]PHYS 225: Win17 and PHYS 324: Sum17. [d]PHYS 225: Sum17, Win18 and PHYS 324: Aut17. [e]PHYS 225: Win18

Overall, students did not do well on either Task 4.2 or Task 4.3. Only roughly half of the students answered correctly that the particles from the two different ensembles are experimentally distinguishable in either context. At most two-fifths of the students gave correct reasoning. Three features stand out from the data in Table 4.25: (1) more students gave correct reasoning in the spin context than in the infinite square well context ($p < 0.001$), (2) the form of complex number (rectangular versus exponential) does not seem to impact student performance ($p > 0.05$), and (3) data from the exam (in spin context) are similar to those (also in spin context) from online surveys ($p > 0.05$), which suggests that our data from online surveys are reliable indication of student performance.

4.4.3 Identification of Student Difficulties

Since the results from tasks Task 4.2 and Task 4.3 do not seem to depend on the form of complex number or administration (on exam or online survey), we discuss in detail student responses from an online survey that used *rectangular* forms in each context as an illustration of student reasoning.

On Task 4.2, about two-fifths of the students incorrectly stated that the particles in ensembles A and B are indistinguishable. The most common incorrect reasoning (given by about 20% of all students) was that the probabilities are the same since a modulus square is used to determine probabilities. Many of these students did not specify whether they were referring to probabilities for the z-component alone or to probabilities for all the components. However, some did talk explicitly about the other components. Overall, less than 5% of all students explicitly stated that the probabilities for measuring all three spin components (x, y, and z) of the spin are the same. We speculate that some of the remaining students who said the states were indistinguishable did not invoke measurements of components other than the z-component (recall the states are written in the S_z basis). We have discussed a similar error in chapter 2, section 2.5.4, that it was very common for students to reason that a state expressed in a particular representation or basis does not provide information about other observables.

About 5% (of all students) reasoned that "the states only differ by a phase", or that imaginary numbers cannot be measured, an idea that has been previously discussed in the research literature [69]. For example, one student wrote: "There is no experiment to distinguish them, because the i will go away when you measure it. Imaginary things cannot be measured." We found this statement very interesting. Although complex numbers are used in quantum mechanics, the eigenvalues (*i.e.,* the results of measurements) and the probabilities are represented by real numbers. Thus, it is correct that 'imaginary things cannot be measured.' However, imaginary numbers in quantum mechanics are necessary to represent differences in how the basis states interfere with one another. "Because the *i* will go away when you measure it" seems to reflect this student's belief or expectation of the role that complex numbers play in superposition states. The results suggest that these students do not understand the purpose of using complex numbers in superposition states. See section 4.6 for results from other questions that were designed specifically to probe student views about complex numbers not only in quantum mechanics but also in electrodynamics.

On Task 4.3 in the context infinite square well, only about half of the students correctly answered that the ensembles are experimentally distinguishable. Less than one-fifth gave correct reasoning. Few (less than 5%) stated that the 'relative phase is different' or that 'one can measure a different basis'. Although these students did not describe what observable could be measured, they did provide a correct explanation of the underlying physics and we categorized their responses as "correct reasoning." About 12% described a correct experiment by specifying the observable(s) to be measured. The experiment they described involved measuring either position or momentum (which is also correct). Interestingly, some students who answered correctly but with incorrect reasoning (12%) stated that the ensembles can be distinguished by measuring along a different direction (e.g., along y direction). It may be that students recognized this question as similar to the one in the spin context but were not able to generalize the idea to a new context.

Many students (24%) incorrectly reasoned that the probabilities are the same since the magnitudes of the coefficients are the same. This response is very similar to the most common incorrect response on Task 4.2. Most of these students did not specify whether they were referring to probabilities for energies alone or to probabilities for all observables. However, a few (about 5%) explicitly stated that the particles have the same probability density. This error is consistent with prior research on student ability to use superposition for position-space wave functions with different relative phases [70].

Although the student answers and reasoning do not seem to depend on the form of complex number, we found that slightly more students used the term "phase" or "relative phase" when the complex coefficients are written in the exponential form. It is likely that students tend not to think spontaneously about the phases of complex numbers when they are written in the rectangular form since phase is typically defined as the argument of a complex exponential.

4.4.4 Summary of Results

The results from this study suggest that many students do not recognize that the relative phases in a superposition state can have measurable effects. For two superposition states that differ only by a relative phase, students often say that that the two states are experimentally indistinguishable since the magnitudes of the coefficients are the same. It seems that many students focus only on the measurements of the observables corresponding to the basis that is used in the problem statement and do not invoke measurements of other observables. This occurs even if students have extensive experience with Stern-Gerlach experiments (including measurements on different components of spin). The errors that students made may not simply be due to a neglect of invoking measurements of different observables, but may potentially reveal insights into student perceptions of the usage and the role of complex numbers in quantum mechanics.

It is worth noting that the difficulties not only occur when the coefficients are complex numbers, but also arise when the coefficients are purely real (with opposite signs) as was true for some versions of the tasks. We observed similar difficulties arose in both spin and infinite square well contexts. However, both sophomores and juniors performed consistently less well on the isomorphic question in infinite square well context. This may be because most textbooks do not explicitly discuss the relevant experiments in the infinite square well or other contexts involving a position-dependent potential. It is also possible that the position-space complex-valued functions are more abstract for students to reason with compared to complex numbers used in the spin context. Nonetheless, it appears students do not transfer from one context to the other.

4.5 Student Ability to Compare Probabilities for Superposition States with Different Relative Phases

As discussed in section 4.4, many students do not recognize that superposition states that differ only in the relative phases between the basis states are experimentally distinguishable. Many students tend to focus only on measurements corresponding to the given basis states (which have identical probabilities) rather than measurements for other observables. We hypothesized that students do not recognize the states experimentally distinguishable simply because they did not invoke measurements for other observables. To test this hypothesis, we conducted a follow-up study in which we explicitly prompted students to compare probabilities for other observables. Thus, students did not need to recognize whether or not the states are experimentally distinguishable, but only needed to be able to decide whether or not the probabilities are different.

The results presented in section 4.4 showed that student reasoning about relative phase on the open-ended questions is context-dependent. They did better in the spin context than the infinite square well context. Therefore, we designed isomorphic tasks in three contexts: (1) spin, (2) infinite square well, and (3) purely mathematical context. The first two are both contexts in quantum mechanics. However, spin has discrete eigenvalues while position has continuous eigenvalues. Using both discrete and continuous cases allows us to identify similarities and differences in student reasoning. Moreover, we can also compare student performance on the tasks with different prompts: open-ended through the questions in chapter 4.3 and "comparing probabilities" through the questions in this chapter. The mathematical context was used to test whether or not student performance differs in a physics context from a purely mathematical context.

We first discuss results from the tasks in a spin context in section 4.5.1. We then discuss student performance on the tasks in the infinite square well context in section 4.5.2. Lastly, results from the tasks in a mathematical context are presented in section 4.5.3.

4.5.1 Spin Context

Figure 4.3 shows two tasks about comparing spin probabilities. The tasks were administered on midterm exams in PHYS 225 after lecture instruction on Stern-Gerlach experiments, operators, and measurements.

Task 4.4 has two parts. Part A asks for the probability of measuring $+\hbar/2$ for S_x for a state written in the S_z basis. In part B, students are asked to compare this probability to the probability of measuring the same quantity for a new state that differs only by a relative phase. In the new state, the relative phase is $\pi/2$ and the complex coefficient is expressed in the rectangular form, i. We choose to use a purely imaginary coefficient because the results from the open-ended questions discussed in section 4.4 suggest that (1) student performances do not seem to vary from purely imaginary coefficient to purely real (but negative) coefficient and (2) student perceptions of the role of complex numbers in quantum mechanics seem to impact their responses.

The setup for Task 4.5 is very similar to Task 4.4. However, there are two major differences: (1) Task 4.5 does not ask for probability of measuring $+\hbar/2$ for S_x for the one of the states as it does in Task 4.4 (2) the complex coefficient is expressed in the exponential form, $e^{i\pi/2}$, rather than rectangular form.

In order to answer Tasks 4.4 and 4.5, students need to know how to determine spin probability and to be able to change basis, as discussed in section 4.4.1. In both Tasks, the probabilities of measuring $+\hbar/2$ for S_x are different for the two states due to different relative phases.

Task 4.4

Consider a spin state $|\beta\rangle = \sqrt{\frac{3}{5}}|+\rangle_z + \sqrt{\frac{2}{5}}|-\rangle_z$.

A. Calculate the probability of measuring spin up in the x-direction.

B. If the state is replaced by $|\beta'\rangle = \sqrt{\frac{3}{5}}|+\rangle_z + i\sqrt{\frac{2}{5}}|-\rangle_z$, will the probability be the *same as* or *different from* the probability you computed above?

Task 4.5

Suppose the particles in ensemble C are all described by the state vector $|\psi_C\rangle = \frac{3}{5}|+\rangle_z + \frac{4}{5}|-\rangle_z$, and that the particles in ensemble D are all described by the state vector $|\psi_D\rangle = \frac{3}{5}|+\rangle_z + \frac{4}{5}e^{i\pi/2}|-\rangle_z$. Would the probability that a particle from ensemble C is measured to have spin up in the x-direction be the *same as* or *different from* the probability that a particle from ensemble D is measured to have spin up in the x-direction? Show your work and/or explain your reasoning.

Figure 4.3. Tasks about relative phase administered on midterm exams in PHYS 225.

4.5.1.a Overall performance on Task 4.4 and Task 4.5

Table 4.26 shows student performance on Tasks 4.4B and 4.5. Student performance on Task 4.4A is not discussed because results from tasks that examine student ability to *compute* spin probability have been presented in chapter 2.

On both Task 4.4B and Task 4.5, about two-thirds of the students gave a correct answer. Only about half of the students provided a correct explanation or computation. Again, student performance does not seem to be affected by the form used to express the complex numbers (rectangular or exponential).

Table 4.26. Student performance on Task 4.4B and 4.5

	Form of complex coefficient	Population	Correct answer	Correct computation/explanation
Task 4.4B	Rectangular	$N = 52$[a]	69%	46%
Task 4.5	Exponential	$N = 64$[b]	64%	44%

[a]PHYS 225: SUM16. [b]PHYS 225: SUM17.

About three-quarters of the students in both academic years attempted to perform a calculation to some extent. Many students used a mixture of qualitative and computational explanations in their responses. Therefore, we do not attempt to categorize student responses simply based on whether or not they performed a computation.

4.5.1.b Comparison of student performance on tasks with different prompts

To determine whether or not student performance on questions involving relative phase depends on the prompt (open-ended or "compare probabilities"), we compare data from tasks (4.4 and 4.5) that ask students to compare spin probabilities and data from Task 4.2, which has an open-ended question in spin context. For Task 4.2, we only use the exam data in PHYS 225 since both Tasks 4.4 and 4.5 were given on exams in PHYS 225. We believe that this comparison is the most straightforward.

Table 4.27. Comparison of student performance on midterm exams

	Prompt	Population	Correct answer	Correct computation/ explanation
Task 4.2	Open-ended	$N = 189$[a]	61%	41%
Task 4.4B Task 4.5	Compare probabilities	$N = 116$[b]	66%	45%

[a]PHYS 225: WIN18. [b]PHYS 225: SUM16 and SUM17.

The data are shown in Table 4.27. The first row in the table shows results from Winter 2018, when students were given an open-ended question and asked whether or not the particles are experimentally distinguishable. The second row has results aggregated from summer quarters, when students were prompted to compare probabilities for a particular component of spin.

We used Fisher's exact test to compare the percentages of correct answer between two groups, and to compare the percentages of correct computation/explanation between two groups. The two-tailed p-values are 0.393 and 0.551, respectively. They suggest that neither the differences in percentages is statistically significant. Therefore, we conclude that in spin context, student performance on the open-ended question is very similar to that when students are prompted to compare probabilities.

The results above seem somewhat surprising. From a physics expert's point of view, it might be regarded that the task with the open-ended question (Task 4.2) is more challenging since it requires students to describe an experiment if they consider the particles distinguishable. The tasks with specific prompt only ask students to compare probabilities. However, neither the percentages of correct answer nor the percentages of correct explanation/computation are statistically significantly different. To gain more insights into student reasoning, a close examination of student responses to Task 4.4B and Task 4.5 is discussed below.

4.5.1.c Identification of student difficulties

We have categorized (aggregated) student responses to Task 4.4B and Task 4.5. We discuss in detail some of the common categories of student responses.

Most (40% of all students) of the students who gave a correct answer with correct explanation computed the modulus square of the inner product corresponding to the probability for each state. A few students (5% of all students) argued that the inner products have different magnitudes without actually computing the probabilities. For example, one student responded, *"Different, because now the magnitude of the inner product is different, and the square of the magnitude of the inner product is the probability.* $\left|\sqrt{\frac{3}{5}} + i\sqrt{\frac{2}{5}}\right| \neq \left|\sqrt{\frac{3}{5}} + \sqrt{\frac{2}{5}}\right|$*."* This type of responses was also considered as providing a correct explanation.

Many (16% of all students) of the students who gave a correct answer with an incorrect explanation made computational errors. The most common (8% of all students) computational error was treating a modulus squared as a "regular" squared. These students used vertical bars in their calculations (not parentheses), but then performed the computation as if they were squaring, with no complex conjugate (a common error discussed in chapter 2). Lastly, they arrived at a complex probability represented by a complex number and concluded that the probabilities are different.

For students who gave an incorrect answer, the most common explanation (16% of all students) was of the form $|a + ib|^2 = |a + b|^2$ (or $\left|a + e^{i\pi/2}b\right|^2 = |a + b|^2$). Some of these students appeared to neglect the cross term of $|a + b|^2$, some correctly computed $|a + b|^2$ but then argued that $|a + ib|^2$ is equal to $|a + b|^2$, while others reasoned that $|i|^2 = 1$. For example, one student answered that *"The probability will be the same as we calculate P [probability] as absolute value of the* $\left|{}_x\langle+|\beta'\rangle\right|^2$*, it will make no difference as* $|i|^2 = 1$*."* All these students recognized that determining probabilities involve inner products, then they *qualitatively* compared the modulus squared of the inner products without explicitly computing them. Their responses appeared to suggest that they do not recognize the role that relative phase (or complex number) plays in superposition states.

Regardless of which answer they gave (that the probabilities are same or different), many students (24%) had difficulty with the modulus square $|a + ib|^2$. They either treated it as a square, $(a + ib)^2$, or confused it with $|a + b|^2$. About 12% of the students seemed to have difficulty with the formalism for determining probabilities. Some did not seem to recognize that determining probabilities involve inner products, others failed to change basis (including students who incorrectly changed basis by multiplying the S_z basis by $1/\sqrt{2}$).

Summary: Comparing student responses to the task involves an open-ended question and responses to the task that prompts students to compare spin probabilities, we have found some similarities and differences. On both tasks, many students did not appear to recognize that the relative phase can affect probabilities. They seemed not to understand the role that complex numbers play in superposition states. On the task that prompts students to compare probabilities, many students attempted computations to some extent, which was not seen on the open-ended question. Many students made computational errors to in their responses.

4.5.2 Infinite Square Well Context

In addition to asking Tasks 4.4 and 4.5 in spin context, we have also administered several tasks prompting students to compare position-space probability density (or probability) in the infinite square well context, as shown in Figure 4.4. The primary goal is to find out whether or not student performance would be different when they are prompted to compare probability densities from student performance on the open-ended question in the same context.

Task 4.6 and Task 4.7 were given as online surveys after lecture instruction on the infinite square well in different academic quarters. Task 4.8 was administered on a midterm exam after lecture instruction and after students had completed a preliminary version of a tutorial intended to help them relate relative phase and probabilities. It is worthwhile to mention that the students were told no tutorial-based questions would be on exams. The questions included on the exams were regarded by the instructor as ones that could be answered based on lecture instruction.

In both Task 4.6 and Task 4.7, students consider two position-space wave functions. In Task 4.6, the wave functions are written in terms of the ground state and second-excited state wave functions. In Task 4.7, the wave functions are not written in the energy basis, but rather as a combination of $\sin\left(\frac{\pi x}{a}\right)$, the ground state wave function, and $\sin^3\left(\frac{\pi x}{a}\right)$. In both questions, students are asked whether or not the position-space probability densities are different and to give an explanation.

In Task 4.8, students were given the initial wave function written as a linear combination of the ground state and first-excited state wave functions at $t = 0$. They were asked to find the probability of the particle being in the left half of the well at a later time. [On the exam, students were also asked to determine other quantities (see Appendix C, Figure C.1 for the full task), such as the expectation value for energy. Thus, additional questions are not shown in the figure.] Students were then asked whether their answers in the previous parts would change if the wave function is instead a wave function with a different relative phase. They were told that no explanation was needed.

Task 4.6

Consider the two wave functions given below for the infinite square well (where N is a constant):
$\psi_A(x) = N(\psi_1(x) + \psi_3(x))$
$\psi_B(x) = N(\psi_1(x) + i\psi_3(x))$
Recall that the energy eigenfunctions for the infinite square well are given by:
$\psi_n(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{n\pi x}{a}\right)$
Would the (position space) probability densities for these two particles be the same or different? Explain your answer.

Task 4.7

Consider the two wave functions given below for the infinite square well (where N is a constant):
$\psi_A(x) = N(\sin\left(\frac{\pi x}{a}\right) + \sin^3\left(\frac{\pi x}{a}\right))$
$\psi_B(x) = N(\sin\left(\frac{\pi x}{a}\right) + i\sin^3\left(\frac{\pi x}{a}\right))$
Would the (position space) probability densities for these two particles be the same or different? Explain your answer.

Task 4.8

A particle is prepared at time $t = 0$ with wave function $\psi_a(x,0) = C(\phi_1(x) + \phi_2(x))$.

- **A.** Find the probability of finding the particle in the left half of the well ($x < L/2$) at any time $t \geq 0$.
- **B.** If the particle were prepared instead with wave function $\psi_b(x,0) = C(\phi_1(x) + i\phi_2(x))$, would your answer to the previous question be different? No explanation needed.

Figure 4.4. Tasks about comparing position probabilities given in PHYS 324.

As explained in section 4.4, the probability densities for the two particles in Task 4.6 (and in Task 4.7) are different due to different relative phases. Similarly, the probability of finding the particle in the left half of the well in Task 4.8 would, in general, be different if the wave function at $t = 0$ is described by a wave function with a different relative phase.

4.5.2.a Overall performance on Tasks 4.6, 4.7, and 4.8

Student performance on Tasks 4.6, 4.7, and 4.8 is shown in Table 4.28. Only about half of the students on Task 4.6 or Task 4.7 correctly answered that the probabilities densities are different. Many students did not explicitly explain how the probabilities densities or the modulus square of wave functions are different. Thus, we do not discuss the prevalence for correct answer with complete reasoning. On Task 4.8, about 62% correctly answered that the position probabilities are different. Recall that the question did not ask for explanation.

Table 4.28. Student performance on Tasks 4.6, 4.7, and 4.8

	Population	Administration	Correct answer
Task 4.6	$N = 174$[a]	Survey	53%
Task 4.7	$N = 87$[b]		55%
Task 4.8	$N = 92$[c]	Exam	62%

[a]PHYS 324: AUT14 and AUT17. [b]PHYS 324: SUM14, AUT15, and SUM16. [c]PHYS 324: AUT16.

Using a Fisher's exact test to compare the percentage of correct answer on exam and that on online surveys (aggregated results from Task 4.6 and Task 4.7), we have found that the two-tailed *p*-values equals 0.181. Therefore, the difference between exam data and survey data are not statistically significant. Although Task 4.8 may be slightly more challenging than Tasks 4.6 and Task 4.7 since it involves time dependence, the result from the statistical test suggests that the data from online surveys are reliable.

4.5.2.b Comparison of student performance on tasks with different prompts

We compare junior students' performances on tasks that are in the infinite square well context with the two different prompts: (1) whether or not the states are experimentally distinguishable, and (2) whether the probability densities are same or different. The data are shown in Table 4.29. (Note that in Autumn 2017, the tasks with different prompts (Task 4.3 and Task 4.7) were given in separate online surveys with one week apart. We assume student answers to these questions are independent and can be used for comparison since students do not receive feedback.)

Table 4.29. Comparison of student performance on online surveys with different prompts

	Prompt	Population	Correct answer
Task 4.3	Open-ended	$N = 119$[a]	48%
Task 4.6 Task 4.7	Compare probability densities	$N = 261$[b]	54%

[a]PHYS 324: SUM17 and AUT17. [b]PHYS 324: SUM14, AUT14, AUT15, SUM16, and AUT17.

Using Fisher's exact test, we found that these two groups of data are not statistically significantly different ($p > 0.05$). That is, student performance on the task with an open-ended question is very similar to that on the tasks that prompt them to compare probability densities. The results are consistent with what found in the spin context (see section 4.5.1b).

4.5.2.c Identification of student difficulties

In order to gain insights into how students reason about position-space probability densities that differ only in a relative phase, we have examined responses from students who incorrectly answered that the probability densities are the same. (Recall that students who gave a correct answer tended not to give thorough explanation for why they decided that the probabilities densities or the modulus square of wave functions are different.)

Many students (about 18% of all students on Task 4.6 and 21% on Task 4.7) either explained that the phase (or the imaginary component) does not affect the probability or stated that the modulus square would get rid of the imaginary component. We saw similar reasoning in the spin context (see section 4.5.1.c). The explanations that these students provided seem to reflect their views or expectations that relative phases or complex numbers would not affect probabilities since a modulus square of a complex number results in a real number. Others (about 16% of all students on Task 4.6 and 18% on Task 4.7) stated that the modulus squares of the wave functions are the same. However, they did not give the expressions for the modulus squares of wave functions.

Although not common, less than 5% of all students on Task 4.6 said that the modulus squares of the coefficients (in the energy basis) are the same. It is likely that these students confused determining energy probabilities with determining position probabilities. It is also possible that they incorrectly used the orthogonality of energy eigenstates, an error that has been discussed in chapter 2. That is, they confused the product of energy eigenfunctions with the inner product of energy eigenstates and then concluded that the cross term (the product of energy eigenfunctions) is zero. We did not notice this type of response on Task 4.7, in which that type of error does not apply.

The results suggest that many students fail to recognize that superposition states with different relative phases have distinct probability densities. Many students seem to believe that (relative) phase does not affect probability density since probability density involves a modulus square. These students do not seem to understand the role of complex numbers play in superposition states. Their beliefs that "phase does not matter" seem to impede student understanding of quantum interference.

4.5.3 Mathematics Context

The results from sections 4.5.1 and 4.5.2 suggest that many students do not recognize that relative phase can affect probabilities when they are prompted to compare spin probabilities or position-space probability densities. Many seem to have a "belief" that the relative phase or complex number does not affect probability or modulus squared. We were interested in probing whether the difficulties are simply due to lack of mathematical proficiency or whether they are influenced by students' beliefs or expectations about the role that complex numbers play in the quantum formalism. In order to answer these questions, we have conducted a further investigation.

Task 4.9, Version 1

Consider two numbers $a + b$ and $a + ib$, where both a are b real and non-zero. Would $|a + b|^2$ be equal to $|a + ib|^2$? (Recall that $|a + ib|^2$ is the modulus squared of $a + ib$.)

Task 4.9, Version 2

Consider two functions below:
$P(x) = f(x) + g(x)$
$Q(x) = f(x) + ig(x)$
where $f(x)$ and $g(x)$ are real functions.

Would $|P(x)|^2$, the modulus square of function $P(x)$ be the same as or different from $|Q(x)|^2$, the modulus square of function $Q(x)$?

Task 4.10

Q1. Consider a function $P(t) = f(t) + g(t)$, where $f(t)$ and $g(t)$ are real functions. What is $|P(t)|^2$, the modulus square of function $P(t)$?

Q2. Now consider another function $Q(t) = f(t) + ig(t)$, where $f(t)$ and $g(t)$ are real functions.
What is $|Q(t)|^2$, the modulus square of function $Q(t)$?

Figure 4.5. Tasks in purely mathematical context given as online survey in PHYS 225.

We designed two tasks in a purely mathematical context, as shown in Figure 4.5. Task 4.9 has two versions. In version 1 (V1), students consider two numbers $a + b$ and $a + ib$, where both a and b are real. Students are asked whether the modulus squared of those two numbers are same or different. Version 2 (V2) asks students whether the functions $P(x) = f(x) + g(x)$ and $Q(x) = f(x) + ig(x)$ are same or different. The two versions are analogous to tasks in spin and infinite square well contexts in which students are asked to compare spin probabilities or position-space probability densities. Task 4.10 has two questions. In Q1, students are asked to determine the modulus square of function $P(t) = f(t) + g(t)$. In Q2, students are asked to determine the modulus square of another function $Q(t) = f(t) + ig(t)$. Again, in each question, students are told both $f(t)$ and $g(t)$ are real. (See section 4.5.1 for the answers to these two tasks.)

Two versions of Task 4.9 were administered parts of the online surveys PHYS 225 at different instructional stages. Task 4.9-V1 was given at the beginning of the academic quarter in Summer 2018, therefore students had not received instruction. Task 4.9-V2 was given in Winter 2018 after students had seen relative phases and time dependence in the spin context, but had not taught position-space wave functions. Task 4.10 were administered as part of an online survey in Winter 2018 after lecture instruction on the infinite square well. The two questions were given on different pages and students were not allowed to change their answers from previous pages.

We administered Task 4.9 given at different instructional stages (before instruction in PHYS 225 and after instruction on spin states relative phase) to assess whether learning of quantum concepts can impact student ability to qualitatively compare the modulus squared of complex numbers of complex-valued functions (that are analogous to wave functions that differ only in a relative phase). Task 10 was used to assess student ability to determine the modulus squares of real and complex-valued functions. These tasks allow us to test whether the primary barrier is a lack of skill in computing modulus squared or having a belief that the relative phase (or complex numbers) does not affect modulus squared.

4.5.3.a Results from Task 4.9

Student performance on the two versions of Task 4.9 at different instructional stages is shown in Table 4.30. For comparison, student performance on the analogous tasks in quantum contexts (spin and infinite square well) is also shown.

Table 4.30. Comparison of student performances in quantum and math contexts

	Context and prompt	Population	Instructional stage	Correct answer
Task 4.9-V1	Compare modulus squared	$N = 49$[A]	Beginning of PHYS 225	82%
Task 4.9-V2	Compare modulus squared	$N = 136$[B]	After instruction on probabilities and relative phase PHYS 225	54%
Task 4.4B Task 4.5	Compare spin probabilities	$N = 116$[C]	After instruction on probability and relative phase PHYS 225	66%
Task 4.6 Task 4.7	Compare position-space probability densities	$N = 261$[D]	After instruction on infinite square well PHYS 324	54%

[A]PHYS 225: SUM18. [B]PHYS 225: WIN18. [C]PHYS 225: SUM16 and SUM17. [D]PHYS 324: SUM14, AUT14, AUT15, SUM16, and AUT17.

On Task 4.9-V1, about 82% of the students in group A answered correctly at the beginning of the course PHYS 225 (students had not been taught quantum mechanics). On Task 4.9-V2, only about 54% of the students in group B answered correctly after lecture instruction on probability and relative phase. The difference in performance is statistically significant ($p < 0.01$). The results suggest that student ability to compare the modulus squared seems to decrease after instruction on quantum mechanics.

Comparing student performance on Task 4.9-V2 and student performance on tasks in the quantum contexts, we found that the results are very similar ($p > 0.05$). This suggests that after lecture instruction, student ability to compare modulus squared is not context-dependent. However, student performance on Task 4.9-V1 is statistically significantly better than student performance on the tasks in the quantum contexts. Recall that students had Task 4.9-V1 before instruction on quantum mechanics, but had Task 4.9-V2 after instruction on probability and relative phase. Again, the results suggest that student ability to compare the modulus squared seems to decrease after instruction on quantum mechanics.

For students in group B who answered incorrectly after instruction, some explained that the imaginary component does not matter since the modulus squared would "get rid" of the imaginary component. Similar to what we have found from results on tasks in spin and infinite square well contexts (see sections 4.5.1 and 4.5.2), many students seem to have beliefs or expectations that the imaginary component would not affect the modulus squared of complex-valued functions. This type of belief or expectation does not only occur in the quantum context, but also arise in the math context. Others explained that two functions have the same modulus squared since the modulus square of i equals to 1. These students seem to have the similar belief.

4.5.3.b Results from Task 4.10

We start with a discussion of student performance on each of the questions in Task 4.10. We then compare student performance on Task 4.9-V2 and that on Task 4.10. Since both tasks were given after instruction on probability and relative phase, this comparison would allow us to decide whether the students simply lack skill in determining modulus squared or students have a "belief" that the relative phase (or the complex number) does not affect modulus squared or probability.

In order to make a direct comparison between results from Task 4.10 and Task 4.9-V2, we eliminated responses that did not explicitly provide the results of a calculation or did not simplify the products. After this screening, 99 responses out of 148 were used. On Q1, about 85% of the students gave essentially a correct answer. This includes answers (about 11% out of 99 responses) with *minor* errors such as missing the factor of 2 in the cross term $2f(t)g(t)$. Responses that did not include the cross term were considered as incorrect. On Q2, about 70% gave essentially a correct answer, including those with minor errors such as having the opposite sign for $g(t)^2$. We regarded responses that included a cross term as incorrect. About 57% were correct (including those with minor errors) on both questions.

Table 4.31. Comparison of student performances on tasks in math context

	Context and prompt	Population	Correct answer
Task 4.9-V2	Compare modulus squared	N =136[a]	54%
Task 4.10	Determine each modulus squared	N = 99[b]	87%

[a]PHYS 225: WIN18. [b]PHYS 225: WIN18.

We compared student answers to each question in Task 4.10 and divided them into two categories: "same modulus squared" and "different modulus squared." We found that 87% of the responses fit into the "different" category. We compare the percentage of category "different" on Task 4.10 and the percentage of correct response on Tasks 4.9, as shown in Table 4.31. The difference in performance is statistically significant ($p < 0.001$). Thus, we can conclude that after instruction, students performed much better when they were asked to determine the modulus squared for each function than when they were asked whether the modulus squared are the same or different.

4.5.4 Summary

Sections 4.4 and 4.5 together discuss an investigation into student understanding of the measurable effects of relative phase. During the investigation, we first designed open-ended questions to examine the extent to which students recognize how relative phase can affect experimental results. The tasks involve both spin and infinite square well contexts. Students are given two quantum states that differ only by a relative phase and they are asked whether the students are experimentally distinguishable. Many students tend to answer that they are not experimentally distinguishable. Many argue that the probabilities are the same or simply that "the phase does not matter." Since the quantum states in the original questions are written in a particular basis, we hypothesized that students did not recognize that the states are experimentally distinguishable simply did not invoke measurements for other observables. To test this hypothesis, we designed new tasks (in both spin and infinite square well contexts) that prompt students to compare the probability for a particular observable (that would result into different values of probabilities). The results on the specific prompt questions were very similar to the open-ended questions (the difference is not statistically significant), therefore, we rejected the hypothesis. We then developed new hypotheses that (1) students simply lack skill in determining the modulus squared of complex numbers, and (2) students have a "belief" that the relative phase does not affect probabilities or modulus squared. Lastly, we designed analogous questions in a mathematical context that ask students to compare or determine the modulus squared of two complex numbers (or complex-valued functions). The results suggest that mathematical proficiency does not seem to be the primary barrier. Rather, students appear to believe that the relative phase does not affect probabilities or modulus squared.

The results suggest that many students do not understand the purpose of using complex coefficients in superposition states. Student views and expectations of the role of complex numbers appears to undermine student understanding of quantum interference.

4.6 Student Views about the Roles of Complex Numbers in Electrodynamics and Quantum Mechanics

Our results from previous sections suggest that student views and expectations of the role of complex numbers in quantum mechanics seem to influence their understanding of quantum interference. In this section, we discuss a follow-up study on student views about the roles that complex numbers play in electrodynamics and quantum mechanics. In electrodynamics, complex exponentials are often used to describe electromagnetic waves for the purpose of mathematical convenience. Thus, the roles of complex numbers in electrodynamics and quantum mechanics are distinct. We thus may obtain a deeper understanding of student ideas through examining complex numbers in physics when students are given two contrasting cases.

We administered a task in online surveys in PHYS 322 in Summer 2017 and Winter 2018 after lecture instruction on electromagnetic waves. Note that at the UW, it is typical that electromagnetic waves are covered at the end of the second of a three-quarter sequence junior-level E&M course, PHYS 322. Most of the students involved in this study had already completed PHYS 324, or at least have completed PHYS 225.

4.6.1 Task Design

The task (Task 4.11) we used to probe student ideas about complex numbers is shown in Figure 4.6. Task 4.11 consists of two questions. In Q1, students consider two different expressions for the electric field of an electromagnetic wave. The expressions are identical except that one is written as a cosine function, and the other is written as a complex exponential. Students are asked for the similarities and differences between the electric fields. They are also asked for the purpose of using complex numbers to describe electromagnetic waves. Because the electric field is a measurable quantity, it is always real. However, a complex exponential is often more convenient for mathematical computations than a sinusoidal function. Thus, the electric field is often written as a complex exponential, but since the imaginary component has no physical meaning, it is dropped when the computations are completed.

In Q2, students consider whether the purpose of using complex numbers in quantum mechanics is the same as the purpose of using complex numbers to describe electromagnetic waves. We were testing whether students recognize that complex numbers are necessary in quantum mechanics, since the relative phases in a superposition state have measurable effects. That is, the relative phases can affect the probabilities. Thus, complex numbers are needed to describe superposition states with different interference results. The question thus probe student thinking about the different purposes of using complex numbers in quantum mechanics and in electrodynamics.

Task 4.11

Q1. Suppose that in a given homework problem you were presented with the electric field of an electromagnetic wave as $\vec{E} = E_o \cos(kz + \omega t)\hat{y}$.
In a separate problem, you were presented with electric field of an electromagnetic wave as $\vec{E} = E_o e^{i(kz+\omega t)}\hat{y}$.
How, if at all, are the electric fields similar? If at all, are they different?

What do you think is the purpose of using complex numbers to describe electromagnetic waves?

Q2. Complex numbers are often used in quantum mechanics as well. Is the purpose of using complex numbers in quantum mechanics *same as* or *different from* the purpose of using complex numbers to describe electromagnetic waves? Explain.

Figure 4.6. A task probes student views about the roles of complex numbers given in PHYS 322.

4.6.2 Student Views of the Role of Complex Numbers in Representing Electromagnetic Waves

There were 161 students who answered Task 4.11. We divided student responses regarding the purpose of using complex numbers to describe electromagnetic waves into three categories according to which students thought it was for: (1) convenience, (2) necessity, or (3) they were not sure. About 44% of the students gave responses consistent with category (1) and another 44% were placed in category (2). The rest of the students (about 12%) were not sure about this question.

Most (37% of all students) of the students in category (1) answered that using complex numbers rather than sinusoidal functions simplifies the mathematical computation. For example, one student answered that: "*It is mathematically convenient to represent it in complex number, taking derivative and integral is easy*." Others stated that complex numbers can describe both amplitude and phase in a single term, or are easier to read or write.

Many students seem to believe that complex numbers are necessary for describing electromagnetic waves (category 2). About 16% (of all students) explained that complex numbers provide more information about an electromagnetic wave since they involve both real and imaginary components. However, these students did not explicitly describe what extra information the imaginary components provide. About 9% (of all students) stated that electromagnetic waves are complex, and complex numbers describe the imaginary components of electric fields. These students did not appear to recognize that a measurable physical quantity must be real.

About 7% (of all students) answered that the imaginary component can be used to describe the magnetic field. For example, *"E and B are perpendicular (real and Im [imaginary] are perpendicular)."* This student correctly stated that the electric and magnetic fields in an electromagnetic wave are perpendicular. It is also true the real and imaginary axes are perpendicular. However, this student failed to consider relative phase between the electric and magnetic fields. The electric filed is in phase with the magnetic field. However, the real component ($\cos(kz + \omega t)$) and imaginary component ($\sin(kz + \omega t)$) are 90 degrees out of phase. Thus, the imaginary component cannot describe the magnetic field.

The results from Q1 in Task 4.11 suggest that many students do not understand the purpose of using complex numbers to describe electromagnetic waves. Many seem to believe that complex numbers are necessary. It is possible that some of them fail to recognize that electric and magnetic fields can only be real. Others appear to believe that the imaginary components contain some important information about the system. However, these students did not specify what information is contained in the imaginary components. We speculate that many students tend to passively accept that complex numbers can be used to describe electromagnetic waves without understanding the purpose or appropriateness.

4.6.3 Student Views of the Roles of Complex Numbers in Electrodynamics and Quantum Mechanics

Question 2 in Task 4.11 probed student thinking about the roles of complex numbers in electrodynamics and quantum mechanics. Since not every student given Task 4.11 had taken quantum mechanics, their responses were dropped. Thus, the sample discussed below contain 147 responses.

Only about one-third (31%) of the students answered correctly that the purposes are different. About 14% (of all students) correctly explained that complex numbers are necessary in quantum mechanics, while they are for mathematical convenience in electrodynamics. For example, one student explained that *"It is different. Complex numbers in quantum mechanics actually contribute to the probability of a certain measurement, while the imaginary term is mostly ignored for dealing with EM waves."* This student correctly recognized that complex numbers can affect probabilities, while the imaginary component is ignored for electromagnetic waves. Other students who also said the purposes are different tended to describe the quantum scenarios in which complex numbers are used (*e.g.,* superposition states) rather than to describe how the purposes are different.

Table 4.32. Student performance on Q2 of Task 4.11

	Same purpose	Different purposes (Correct answer)	Correct answer and explanation	Others
N =147[a]	64%	31%	14%	5%

[a]PHYS 322: SUM17, WIN18.

About two-thirds of the students answered correctly that the purposes are the same. Some (about 16%) stated that in both cases, the purpose is “to make math easier.” Others (about 28%) explained that the wave equation is applied in both electrodynamics and quantum mechanics, or that electromagnetic waves follow quantum mechanics rules. These students seem to believe that complex numbers are necessary in both quantum mechanics and electrodynamics. Some students explained that complex numbers are needed to describe the phase of an electromagnetic wave. Another 14% did not provide explanations.

The results suggest that many students do not recognize the that purpose of using complex numbers in quantum mechanics is different from that in electrodynamics. Many students tend to believe that the purposes are the same. The explanations students provided suggest that they tend to use complex numbers in quantum mechanics and electrodynamics as mechanical procedures without understanding the purposes.

4.7 Summary

In this chapter, we have discussed our investigation into student understanding of relative phases in superposition states. This investigation involves three aspects: (1) student understanding of the mathematical behavior of complex exponentials, (2) student ability to recognize that relative phases can affect probabilities in quantum mechanics, and (3) student ideas about the purposes of using complex numbers in quantum mechanics and electrodynamics. We found that many students do not recognize that relative phase can have measurable effects. Many do not recognize the purpose of using complex numbers in superposition states (in quantum mechanics) or the purpose of using complex numbers in electromagnetic waves.

When probing student ability to recognize that relative phase can have measurable effects, we used questions in three different scenarios: (1) open-ended questions that ask whether or not the superposition states that differ only by a relative phase are experimental distinguishable, (2) questions that prompt students to compare probabilities, and (3) analogous questions in mathematical context that ask students to compare or determine the modulus squares. The first two scenarios involve both spin and infinite square well contexts.

On the open-ended questions, many students appeared to focus on the probabilities for the observable corresponding to the basis states in which the state vector is written in, and did not seem to invoke measurements for other observables. However, both the textbook and instructor in the sophomore course covered various Stern-Gerlach experiments and the significance of relative phases. This suggests that many students do not recognize that relative phases can affect probabilities and the role that complex numbers play in superposition states.

When students were asked to compare probabilities for superposition states that differ only by a relative phase, the prevalence of correct answer is very similar to that for the open-ended questions. Moreover, student performance was also similar to that on the analogous mathematical question that prompted students to compare modulus squared. In contrast, when asked to determine the modulus squared for each function, student performed much better than they did for comparing modulus squared. The results suggest that failure to recognize that relative phases can affect probabilities does not simply result from a lack of mathematical proficiency. Interestingly, many students seem to have a belief or expectation that phases or complex numbers do not matter for determining probabilities or modulus squares. It seems that this belief or expectation hinders student understanding of relative phases.

To gain further understanding of student beliefs about using complex numbers in physics, we examined student views about the purposes of using complex numbers in electrodynamics and in quantum mechanics. We found that many students do not recognize that complex numbers are used to describe electromagnetic waves for mathematical convenience. Moreover, many students seem to believe that the purposes of using complex numbers in quantum mechanics and in electrodynamics are the same, yet not many provided complete explanations. It is likely that students tend to view the usage of complex numbers as mechanical procedures and has not grasped the coherence between the mathematical formalism and the physical world.

Our findings provide implications for instruction. For quantum interference in particular, the rationale or purpose of using complex numbers to describe quantum states needs to be made clear to students. This may help students build a motivation to deal with complex numbers and potentially enhance their ability to make sense of the relevant formalism in quantum mechanics. To give a broader perspective, we argue that teaching separately physics concepts and mathematical procedures is not effective. Instructions should integrate these two aspects and bridge the gap between reasoning about the physics and doing the math. The goal is to help students make meaning of the mathematical formalism and understand how the formalism is consistent with real-world physical phenomena.

Chapter 5. Development and Assessment of Tutorials on Relative Phase

The results discussed in chapter 4 suggest that many students do not relate relative phase in superposition states to experimental results. In particular, they do not recognize that relative phases can affect the probabilities of measurement outcomes. Although some students lack skill in calculating the modulus squared of complex numbers (one of the steps required to determine probabilities), mathematical proficiency does not seem to be the primary barrier. Many students seem to have a belief that the relative phase does not matter. The results of our investigation motivated us to develop curriculum to help students make sense of the mathematical expressions in quantum mechanics.

To help students recognize the measurable effects of relative phases, we have developed a new tutorial *Quantum interference with spin states* and modified two existing tutorials *Superposition in quantum mechanics* and *Time dependence in quantum mechanics.*

In section 5.1, we discuss the development and assessment of the new tutorial *Quantum interference with spin states,* which illustrates not only how research can drive curriculum development, but also how curriculum development can, in turn, drive investigations into student understanding. In section 5.2, the modifications and assessment of the tutorial *Superposition in quantum mechanics* are documented. Lastly, the modification of the tutorial *Time dependence in quantum mechanics* is discussed in section 5.3.

5.1 Development and Assessment of the Tutorial *Quantum Interference with Spin States:* An Example of How Curriculum Development Can Drive an Investigation into Student Understanding

As discussed in chapter 1, research by the UW PEG involving curriculum development is an iterative process. Three components, investigations of student understanding, development of curriculum, and assessments of curriculum interact with and enhance each other. In this section, we present a case in which curriculum development has played an important role in driving an investigation of student understanding.

At an early stage of the investigation into student ability to recognize the measurable effects of relative phase, we found that many students do not recognize that relative phases or complex numbers are used to (and needed to) describe superposition states. We decided to develop a tutorial, *Quantum interference with spin states*, which makes use of a scientific inquiry approach. By this we mean that the tutorial is designed to have students start with developing familiarity with physical phenomena. Students are then guided to write possible expressions for the quantum states that are consistent with the experimental results. We thought this approach might be productive since it emphasizes for students the necessity of complex numbers and they could come to recognize this for need themselves. Moreover, this approach is consistent with how the textbook and lectures in PHYS 225 (Introduction to quantum mechanics) introduce relative phase.

In this section, we first describe the tutorial *Quantum interference with spin states* in section 5.1.1. We then present the results from a first assessment in section 5.1.2. The lack of positive results from that assessment motivated further investigation into student ability to infer quantum states from experimental results. This second investigation is presented in section 5.1.3.

5.1.1 Development of New Tutorial *Quantum Interference with Spin States*

The tutorial *Quantum interference with spin states* is intended to help students not only recognize the measurable effects of relative phases, but also to help develop skill in inductive reasoning and interpret the mathematical formalism of quantum mechanics. In particular, students are guided to infer mathematical expressions for quantum states based on experimental results. They are expected to first recognize the need for using complex numbers for coefficients and then to determine the complex coefficients based on the statistical experimental results. This instructional approach is consistent with how relative phase is introduced in McIntyre's textbook [54] and many others.

5.1.1.a Tutorial *Quantum Interference with Spin States*

In the first section of the tutorial (see Appendix A, Figure A.10), students consider an ensemble (ensemble A) of spin-1/2 particles. Each particle in ensemble A can be described by the state vector $|\psi_A\rangle$. They are told that when a thousand of these particles are sent through a Stern-Gerlach apparatus SG$_x$, with a non-uniform magnetic field oriented along the x-axis, half of the particles are measured to have spin up. Students are asked to write a possible expression for $|\psi_A\rangle$ in the S_x basis. The expectation is that most will write $|\psi_A\rangle = \frac{1}{\sqrt{2}}|+\rangle_x + \frac{1}{\sqrt{2}}|-\rangle_x$ (although there are many other possibilities). Students are then prompted to consider whether or not there is more than one state vector that can describe the particles in ensemble A. At this point, most are expected to recognize that the coefficients can be either positive or negative.

Students then consider ensemble B, which has the same experimental results as for ensemble A when the particles are sent through SG$_x$. However, students are told that ensemble A and ensemble B have different results when the particles are sent through a Stern-Gerlach apparatus

SG$_z$, with a non-uniform magnetic field oriented along the z-axis. All the particles from ensemble A are measured to have spin up, while all the particles from ensemble B are measured to have spin down. Students are then asked to modify their expressions for state vectors $|\psi_A\rangle$ and $|\psi_B\rangle$ based on the additional experimental results. At this point, students are expected to write $|\psi_A\rangle = \frac{1}{\sqrt{2}}|+\rangle_x + \frac{1}{\sqrt{2}}|-\rangle_x$ and $|\psi_B\rangle = \frac{1}{\sqrt{2}}|+\rangle_x - \frac{1}{\sqrt{2}}|-\rangle_x$.

At the end of the first section of the tutorial, students are given the expression of another state vector $|\phi\rangle$ that differs from $|\psi_B\rangle$ only by an overall factor of i. They are asked to rewrite this state vector in terms of $|\psi_B\rangle$, and then are asked whether or not there is an experiment that can be used to distinguish between the states $|\phi\rangle$ and $|\psi_B\rangle$. Students are expected to recognize that an overall phase factor does not have measurable effects and work through the mathematics that demonstrates their answer.

In the second section of the tutorial, students consider ensemble C with each of the particles prepared in state $|\psi_C\rangle$. The particles in ensemble C have the same results as particles in ensembles A and B when the particles are sent through SG$_x$. However, when the particles in ensemble C are sent through SG$_z$, three-quarters of the particles are measured to have spin up and one-quarter are measured to have spin down. Students are first prompted to think about whether they expect the coefficients for state $|\psi_C\rangle$ would be the same as the coefficients for either $|\psi_A\rangle$ or $|\psi_B\rangle$. They are then instructed to write each of the coefficients as a positive real number multiplied by a complex exponential (i.e., $re^{i\theta}$). Then, they are guided to determine both the magnitudes and the phases of the coefficients through a series of questions. First, they are asked to determine the magnitudes of the coefficients based on the results from SG$_x$ experiment. Then, they are asked to write $\left|{}_z\langle +|\psi_C\rangle\right|^2$ in terms of the phase of each coefficient. Lastly, they are guided to equate their expression with the experimental results from SG$_z$ and to determine the phases.

At the end of the tutorial, students are given an opportunity to reflect on what they have done. They are expected to generalize the idea that relative phase can affect experimental results.

5.1.1.b Tutorial homework *Quantum Interference with Spin States*

The homework (see Appendix B, Figure B.5) is intended for students to reinforce the idea developed in tutorial that the overall phase does not affect probabilities but the relative phase does. Students are given a quantum state written in the S_z basis with two unknown coefficients c_1 and c_2. They are asked to calculate the probability that the particle is measured to have spin up in the x-direction. They are then asked to find the largest and smallest values of this probability in terms of coefficients c_1 and c_2, as well as the corresponding conditions. Finally, students are given a quantum state that differs only by an overall phase, and they are asked to show whether or not the probability for S_x would be different from that in the original state.

5.1.2 Preliminary Assessment of the Tutorial *Quantum Interference with Spin States*

In this section, we discuss the implementation of the tutorial *Quantum interference with spin states* and provide results from preliminary assessment. For the assessment, we report results from both informal in-class observation and formal online surveys.

The tutorial *Quantum interference with spin states* was implemented in PHYS 225 in Winter 2017. Students worked through the tutorial after lecture instruction on Stern-Gerlach Experiments, operators, and measurements. In the lectures and textbook, students were shown how to use Stern-Gerlach experiments to infer the coefficients of quantum states and they were taught that relative phase can affect probabilities.

Informal observation: We observed the class while students were working through the tutorial. We noticed that students did well until they reached the part in which they were asked to write an expression for state $|\psi_C\rangle$ from the experimental results. Many did not seem to recognize that the coefficients for state $|\psi_C\rangle$ must be different from the coefficients for $|\psi_A\rangle$ and $|\psi_B\rangle$. Moreover, many students struggled with the mathematical procedures that require using the results of the modulus squared of complex exponentials to solve for the phases. As a result, many students did not finish the tutorial in class. (They were told to work through the remainder of the tutorial on their own before doing the homework.)

Formal assessment: Task 4.7 (see Figure 4.2 in chapter 4) was used to assess the effectiveness of the tutorial. Recall that in Task 4.7, students were given two spin states that differ only by a relative phase and they were asked whether or not the states were experimentally distinguishable. Task 4.7 was given to students as an online pretest about two weeks before students completed the tutorial. It was then given again to the same students as an online post-test about three weeks after they had completed the tutorial. (The pretest results were discussed earlier in chapter 4, section 4.4.) The students had also studied time dependence and the infinite square well when they took the post-test, but not the pretest.

Table 5.33. Student performance on Task 4.7 before and after tutorial instruction

	Pretest (N = 126[a])	Post-test (N = 124[a])
Correct answer	66%	69%
With correct reasoning	43%	48%

[a]PHYS 225: WIN17.

Student performance on the pretest and post-test is shown in Table 5.33. The prevalence of students who gave a correct answer with or without correct reasoning is similar on both. It is possible that the lack of gain was due to many students not having finished the tutorial in class.

However, the results seem to suggest that student ability to recognize the measurable effects of relative phase, at least in this context, was not improved after the implementation of the tutorial. We conducted a further investigation, discussed in the following section.

5.1.3 Further Investigation into Student Ability to Infer Quantum States from Experimental Results

The results from the formal and informal assessments discussed in the previous section suggest that the ideas developed in the tutorial *Quantum interference with spin states* are challenging for many sophomore-level students. We recognized that this tutorial requires both sophisticated inductive reasoning and a relatively high level of mathematical proficiency with complex exponentials. The findings suggested that many students do not seem to have the necessary level of proficiency. Another prerequisite for completing this tutorial is that students need to be able to recognize that the coefficients for the states can be (and, in general, need to be) complex numbers. In order to test whether or not this idea was presenting a barrier to student learning, we designed two tasks, Task 5.1 and Task 5.2, as shown in Figure 5.1. The tasks were used to test whether or not students would use complex coefficients to describe quantum states when statistical measurement results were given. They were given on midterm exams in classes that were different from the class in which the tutorial *Quantum interference with spin states* was implemented. Task 5.1 was used in Summer 2017 and Task 5.2 was given in Winter 2018.

On Task 5.1, students consider two ensembles (A and B) of identical spin-1/2 particles. All particles in ensemble A are measured to have spin up in the *z*-direction, while three-quarters of the particles in ensemble B are measured to have spin up in the *z*-direction and one quarter are measured to have spin down in the *z*-direction. Students are asked to write possible expressions for the quantum states for the particles in ensembles A and B, respectively. They are also asked, for each ensemble, whether more than one expression can be used to describe the state of the particles. Task 5.2 has similar questions in the context of spin-1 particles.

Task 5.1

Consider ensembles A and B, of identical quantum spin-1/2 particles. Each ensemble contains ten thousand particles.

Let $|\psi_A\rangle$ represent the spin state of each particle in ensemble A. Let $|\psi_B\rangle$ represent the spin state of each particle in ensemble B. Write a possible expression for $|\psi_A\rangle$ and $|\psi_B\rangle$, respectively, based on the experimental results described below.

A. All of the particles in ensemble A are measured to have spin up in the *z*-direction.

B. Three-quarters of the particles in ensemble B are measured to have spin up in the *z*-direction, and one quarter are measured to have spin down in the *z*-direction.

C. Is there more than one state that can possibly describe the particles in ensemble A? Is there more than one state that can possibly describe the particles in ensemble B? Briefly explain your answer.

Task 5.2

Consider ensembles C and D, each is a pure ensemble of ten thousand identical spin-1 particles. Let $|\psi_C\rangle$ represent the spin state of each particle in ensemble C. Let $|\psi_D\rangle$ represent the spin state of each particle in ensemble D.

Suppose the *z*-component of spin is measured. Recall that the eigenvalues of the operator S_z for spin-1 particles are $1\hbar$, $0\hbar$, and $-1\hbar$.

A. Suppose all of the particles in ensemble C have an outcome of $1\hbar$. Write a possible expression for $|\psi_C\rangle$. Briefly explain.

B. Is there more than one state vector that can possibly describe the particles in ensemble C? Briefly explain.

C. For ensemble D, suppose one-quarter of the particles have an outcome of $1\hbar$, one-quarter of the particles have an outcome of $0\hbar$, and one-half have an outcome of $-1\hbar$. Write a possible expression for $|\psi_D\rangle$. Briefly explain.

D. Is there more than one state vector that can possibly describe the particles in ensemble D? Briefly explain.

Figure 5.1. Tasks about inferring quantum states from statistical measurement results in spin contexts

On Task 5.1, since all of the particles in ensemble A are measured to have spin up in the *z*-direction, the state of particles in ensemble A can be represented by $|+\rangle_z$. An overall phase of a state vector does not have any measurable effect, so the state can also be represented by $e^{i\theta}|+\rangle_z$, where θ is an arbitrary angle. For ensemble B, three-quarters of the particles are measured to have spin up in the z-direction, and one quarter are measured to have spin down in the z-direction. We thus can have $|\psi_B\rangle = \frac{\sqrt{3}}{2}e^{i\alpha}|+\rangle_z + \frac{1}{2}e^{i\beta}|-\rangle_z$ (with α and β arbitrary). Students were not required to write the generic forms for the states. Their answers were considered correct if they wrote a possible expression for each state. Similarly, on Task 5.2, one can write $|\psi_C\rangle = e^{i\gamma}|1\rangle_z$ and $|\psi_D\rangle = \frac{1}{2}e^{i\delta_1}|1\rangle_z + \frac{1}{2}e^{i\delta_2}|0\rangle_z + \frac{1}{\sqrt{2}}e^{i\delta_3}|-1\rangle_z$ (with δ_i arbitrary).

5.1.3.a Analysis of student performance on Task 5.1 and Task 5.2

Student performance on Task 5.1 and Task 5.2 are shown in Table 5.34. For each task, more students gave a correct expression for the spin eigenstate (81% on Task 5.1 and 71% on Task 5.2) than for the superposition state (55% on Task 5.1 and 61% on Task 5.2). For the eigenstates, some students (3% on Task 5.1 and 7% on Task 5.2) wrote the state vector multiplied by the eigenvalue (e.g., $\hbar|1\rangle_z$). For the superposition states, it was common (23% on Task 5.1 and 9% on Task 5.2) for students to use the probabilities for the coefficients. For example, many students incorrect wrote $|\psi_B\rangle = \frac{3}{4}|+\rangle_z + \frac{1}{4}|-\rangle_z$. They seemed to confuse the probabilities with the coefficients. Although not incorrect, we noticed that most students used real and positive numbers for the coefficients. It was very uncommon for students to spontaneously use imaginary numbers or negative real numbers whether or not they answered correctly. This is consistent with what Michelini and Zuccarini found among junior-level students in Italy [15].

Table 5.34. Student performance on Task 5.1 and Task 5.2

	Task 5.1 ($N = 64^a$)	Task 5.2 ($N = 189^b$)
Correct expression for the eigenstate (ensemble A or C)	81%	71%
Correct expression for the superposition state (ensemble B or D)	55%	61%
More than one possible state vector for ensemble A/C (with correct reasoning)	55% (19%)	59% (26%)
More than one possible state vector for ensemble B/D (with correct reasoning)	81% (33%)	69% (31%)

[a]PHYS 225: SUM17. [b]PHYS 225: WIN18.

When students were prompted to consider whether or not there is more than one possible state vector for each ensemble, more than half of the students (about 55%-80%) answered yes (which is the correct answer). We noticed that fewer students recognized that there is more than one possible state vector for ensemble A or C (for which there is only one measurement outcome). Many students (about 16% on Task 5.1 and 25% on Task 5.2) explained that there is only one measurement outcome and that therefore only one state vector can describe the particles.

Less than one-third of the students stated that there is more than one possible state vector because the coefficients can be negative real numbers or complex numbers. It seemed that many students did not often recognize that the coefficients can be complex numbers as long as the modulus squared are equal to the probabilities.

The most common (about 30% on Task 5.1 and 13% on Task 5.2) incorrect line of reasoning was that there is more than one possible state vector because one can express the state in different basis. Although the expressions in different basis have different appearances (different coefficients and different basis states, e.g., expressions $\frac{3}{5}|+\rangle_x + \frac{4}{5}|-\rangle_x$ and $\frac{7}{5\sqrt{2}}|+\rangle_z - \frac{1}{5\sqrt{2}}|-\rangle_z$), the expressions are mathematically equivalent (i.e., one can use an equal sign to relate the expressions). These students seem to confuse different state vectors with states written in different bases.

On parts B and D of Task 5.2, about 5% of the students explained that there is more than one possible state vector because one can multiply the state by a constant. States that differ only by an overall factor are not mathematically equivalent, but the overall factors do not have physical meaning. This is because state vectors need to be normalized so that the total probability is equal to one.

About 9% of the students on part D of Task 5.2 explained that there are multiple outcomes and therefore there is more than one possible state. It was possible that some students misinterpreted the question. However, one student explicitly stated that one can break down the state vector into three state vectors, thus there is more than one possible state. The student seemed to think that one (basis) state is associated with one outcome, thus multiple outcomes can be associated with multiple (basis) states. This student appeared to confuse different basis states needed with different superposition states.

5.1.3.b Patterns of student responses and instructional implications

Based on student explanations for why they thought there is more than one possible state vector to describe the experimental results, we identified three common cases (including the correct one) of student interpretation of different possible quantum states with the same experimental outcomes of *one* observable (e.g., in Task 5.1 and Task 5.2, only the measurement outcomes of S_z are given). The results seem to suggest that many students do not recognize when states are mathematically equivalent (i.e., one can use an equal sign to relate the expressions) and when they are physically equivalent (i.e., cannot be distinguished experimentally). In Table 5.35, we categorize, from a physics expert's point of view, whether the state vectors are mathematically equivalent and/or physically equivalent.

Table 5.35. Categorizing whether states are mathematically equivalent or physically equivalent

	Mathematically equivalent	Physically equivalent
States written in different basis e.g., $\frac{3}{5}\|+\rangle_x + \frac{4}{5}\|-\rangle_x$ and $\frac{7}{5\sqrt{2}}\|+\rangle_z - \frac{1}{5\sqrt{2}}\|-\rangle_z$	Yes	Yes
States with different overall factors e.g., $\frac{3}{5}\|+\rangle_x + \frac{4}{5}\|-\rangle_x$ and $3\|+\rangle_x + 4\|-\rangle_x$	No	Yes*
States with different relative phases e.g., $\frac{3}{5}\|+\rangle_x + \frac{4}{5}\|-\rangle_x$ and $\frac{3}{5}\|+\rangle_x + \frac{i4}{5}\|-\rangle_x$	No	No

*Strictly speaking, the second state is not normalized, so it is not a proper state vector.

We discuss the three cases below:

(1) Any quantum states can be expressed in different bases. The expressions in those different bases are mathematically equivalent (i.e., one can use an equal sign to relate the expressions), although they may appear quite different. For example, a quantum state written in the S_x basis $|\psi\rangle = \frac{3}{5}|+\rangle_x + \frac{4}{5}|-\rangle_x$, can also be written in the S_z basis. One can write $|\psi\rangle = \frac{3}{5}|+\rangle_x + \frac{4}{5}|-\rangle_x = \frac{7}{5\sqrt{2}}|+\rangle_z - \frac{1}{5\sqrt{2}}|-\rangle_z$ since there exists a linear relationship between the S_x basis and the S_z basis. These two expressions represent the same state and are experimentally indistinguishable.

(2) States with different overall factors (e.g., $\frac{3}{5}|+\rangle_x + \frac{4}{5}|-\rangle_x$ and $3|+\rangle_x + 4|-\rangle_x$) are not equivalent mathematically since they have different magnitudes. Strictly speaking, the latter is not normalized and it thus is not a proper quantum state. However, these two expressions do not represent different quantum states that can be distinguished experimentally.

(3) The last case in the table involves states that differ by a relative phase. In this case, the states are both mathematically and physically inequivalent.

These different cases, which we identified from student responses, provide us insights into potential instructional strategies. We propose that one can design exercises with contrasting cases, a technique that has been shown effective at improving student learning [71,72]. In the exercises, students can be guided compare and contrast specific examples to evaluate the relationship between the emphasized features (e.g., different basis or different relative phases) and the target concepts (e.g., quantum interference). In particular, students can be prompted to consider whether the states are mathematically equivalent and/or have different probabilities for a particular experimental setup. We are planning to integrate “contrasting cases” into the existing tutorial and assess it in the future.

5.2 Modification and Assessment of Tutorial *Superposition in Quantum Mechanics*

The results from section 5.1 suggest that the ideas developed in the tutorial *Quantum interference with spin states* are challenging for many sophomore-level students. In particular, many students do not seem to be able to infer quantum state vectors from experimental results as required in the tutorial. Therefore, we decided to make use of a different approach that requires the inverse reasoning. We modified another tutorial *Superposition in quantum mechanics* such that students are guided to predict experimental results based on specified quantum states and to infer the relationship between relative phase and probabilities.

This section starts with a description of the original version of the tutorial *Superposition in quantum mechanics* in section 5.2.1. This is followed by a description of the modified tutorial in section 5.2.2. The assessment of the modified tutorial is discussed in section 5.2.3.

5.2.1 Original Tutorial *Superposition in Quantum Mechanics*

The original tutorial (see Appendix A, Figure A. 11) has two sections. The first section is intended to help students recognize that superposition states and mixed states are experimentally distinguishable. The second section helps students use boundary conditions to identify valid wave functions for a particle in the infinite square well potential.

The first section starts with a statement about a wave function that describes a particle in a superposition state in the infinite square well potential. "The wave function given by $\psi(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state." Students are asked whether they agree or disagree with the statement. This statement is used to elicit the incorrect ideas about superposition and mixed states. Students are then guided to answer a sequence of questions that are intended to help them recognize experimental results differ between superposition and mixed states. First, students consider an ensemble of mixed states. They are told that half of the particles are in the ground state and the other half are in the first-excited state. They are asked to sketch the possible position-space wave functions for a particle in this ensemble. They are then asked to compare the probability that a particle is found to be in the left half of the well and the probability that a particle is found to be in the right half. Students are expected to recognize that the probabilities are the same. Next, students consider an ensemble of superposition states. The wave function of each particle is $\psi(x) = \frac{1}{\sqrt{2}}\varphi_1(x) + \frac{1}{\sqrt{2}}\varphi_2(x)$. They are then asked sequence of questions about position probabilities similar to the sequence they were asked for the mixed states. Students are expected to recognize that, for the superposition state, the probability of finding the particle in the left half is greater than the probability of finding the particle in the right half. Therefore, the position probabilities are different for particles in these different ensembles. At the end of the first section, they are given an opportunity to revisit the statement and to reflect on what they have learned from this section. They are expected to recognize that the statement is incorrect since the two ensembles of particles can be experimentally distinguished by measuring position.

In the second section, students are given the same algebraic wave function for the superposition state discussed in the first section. They are asked whether or not the state is an eigenstate of the Hamiltonian and whether or not this wave function can describe a particle in the infinite square well. Then, they are shown two graphical wave functions, one of them vanishes at the boundaries while the other one has a non-zero value at the boundary. Students are asked whether either can represent a particle in the infinite square well. Students are expected to recognize that the probability of finding the particle outside of the infinite square well is zero, therefore the wave function must vanish at the boundaries.

5.2.2 Modified Tutorial *Superposition in Quantum Mechanics*

The results from chapter 4 suggest that many students do not recognize that relative phase has measurable effects. The original version of the tutorial *Superposition in quantum mechanics* did not address this issue directly. In order to help students recognize the role that relative phases or complex numbers play in superposition states, we modified the second section of the tutorial *Superposition in quantum mechanics* substantially (see Appendix A, Figure A.14 for the modified version). For length reasons, this decision necessitated the removal of the second section of the original tutorial that is intended to help students use boundary conditions to determine valid wave functions. We are planning to use that as a homework exercise. In addition, a minor modification was made on the first section based on classroom observations.

In the new version of the second section, students are given the wave functions of two ensembles. Each wave function represents a superposition of the ground state and the second-excited state of the infinite square well. Those superposition states only differ by a relative phase, one has a relative phase of zero and the other has $\pi/2$. Students are asked to sketch the wave function and the probability density for a particle in each ensemble. They are then asked to consider the probability that a particle is found to be in the left half of the well. Students are expected to recognize that the probability densities are different: one of the probability density graph have a maximum at the center and the other have a value of zero at the center. Students are expected to recognize that the ensembles can be experimentally distinguished by measuring position. Then students are given another a superposition state with a different relative phase, π.

They are asked whether this state is distinguishable from those two they have considered earlier. Finally, students are guided to generalize the idea that the relative phase can be experimentally determined.

5.2.3 Assessment of Modified Tutorial *Superposition in Quantum Mechanics*

This section discusses an assessment of the modified tutorial *Superposition in quantum mechanics.* To do so, we designed tasks two types of tasks. Some involve discrete eigenvalues and others involve continuous eigenvalues. The assessment was done in three classes of PHYS 225: class A (Winter 2017), class B (Summer 2017), and class C (Winter 2018).

In 5.2.3.a, we report results from assessment on the tasks with discrete cases in classes A and B. In 5.2.3.b, results assessment on the tasks with continuous cases are discussed in classes B and C.

5.2.3.a Tasks with discrete cases

To assess the effectiveness of the modified tutorial on tasks with discrete cases, we used a task that had been given previously, Task 4.2 (see Figure 4.2) as pretest and a new task, Task 5.3 (see Figure 5.2) as post-test. Task 4.2 has an open-ended question in the context of spin-1/2, while Task 5.3 has an open-ended question in a hypothetical 2-state context, which we believe is more abstract and challenging. Recall that Task 4.2 was given as part of an online survey after relevant lecture instruction in many classes, including class A (Winter 2017) and class B (Summer 2017). Variants of Task 5.3 were administered on the final exams in class A and class B. The variants differ only by coefficients of quantum states.

Task 5.3

Consider a quantum mechanical mouse. You can inquire about the weight of the mouse (using the operator $\widehat{W}$) and the mood of the mouse (using the operator $\widehat{M}$). There are only two possibilities for the weight, the two possibilities for the mood.

The eigenequations for the weight are: $\widehat{W}|light\rangle = 1|light\rangle$, and $\widehat{W}|heavy\rangle = 5|heavy\rangle$. The eigenequations for mood are: $\widehat{M}|happy\rangle = 1|happy\rangle$ and $\widehat{M}|sad\rangle = -1|sad\rangle$. Assume that the eigenstates of $\widehat{M}$ are orthonormal, and the eigenstates of $\widehat{W}$ are orthonormal.

Consider ensemble A, an ensemble of identical copies of the mouse. Each mouse is described by the state $|\phi\rangle = \frac{\sqrt{3}}{2}|happy\rangle - \frac{i}{2}|sad\rangle$. Suppose you measured the weight of each mouse and determined that every mouse had a weight of 1 (i.e., "light").

1. Determine the value of the inner product between a "happy" mouse and a "light" mouse? Explain how you arrived at your answer.
2. Now consider ensemble B, an ensemble of identical copies of the mouse, with each mouse described by the state $|\chi\rangle = \frac{\sqrt{3}}{2}|happy\rangle - \frac{1}{2}|sad\rangle$. Is there an experiment that can be used to distinguish the mice in ensemble A from the mice in ensemble B? If so, describe the experiment. If not, explain why not.

Figure 5.2. A task involving hypothetical case given on final exams in PHYS 225.

Task 5.3 describes a hypothetical scenario of a quantum mouse (the same scenario as discussed in chapter 2, section 2.4). Two quantities can be observed on the quantum mouse: mood and weight. Each observable has two eigenvalues. Students are given the quantum state $|\phi\rangle$ that can be used to describe every mouse in ensemble A. State $|\phi\rangle$ is written in terms of the eigenstates of the mood. Students are told that the weight of each mouse is measured to be 1 (i.e., "light"). In the first question, students are asked to determine the inner product between a "happy" mouse and a "light" mouse. In question two, students are given the quantum state for every mouse in ensemble B, $|\chi\rangle$ also in term of the eigenstates of the mood. $|\chi\rangle$ differs only by a relative phase from the state for ensemble A, $|\phi\rangle$. Students are asked whether the mice from ensemble A are experimentally distinguishable from the mice from ensemble B. (Recall that the first question and the results were discussed in chapter 2, section 2.4. In this section, we focus on the second question.)

To answer, students need to recognize that the state $|\phi\rangle$ is an eigenstate of weight, since each mouse has a weight of 1. Thus, the state $|\phi\rangle$ is equal to state $|light\rangle$ (or $|\phi\rangle = e^{i\theta}|light\rangle$, where θ is an arbitrary angle). State $|\chi\rangle$ differs from state $|\phi\rangle$ by a relative phase. $|\chi\rangle$ is not orthogonal to $|\phi\rangle$ since $\langle\chi|\phi\rangle$ is not zero. Therefore, state $|\chi\rangle$ is not an eigenstate of weight. If one measures the weight, the mice from ensemble A would all have a "light" weight and the mice from ensemble B would have "light" and "heavy" weights. The probability for a mouse in ensemble to be measured to have a "light" weight can be determined by $|\langle\phi|\chi\rangle|^2$ (since $|\phi\rangle$ is the light eigenstate).

We think that the post-test question in the mouse context is more challenging than the pre-test in the spin context for two reasons. First, the pretest question is in a context that students are familiar with while the post-test question is in a new the context. Second, for the spin context, the linear relationships between the basis states are introduced in class. However, for the mouse context, the relationships between the mood eigenstates and the weight eigenstates are not explicitly given in the question, thus students have to recognize that $|\phi\rangle$ (written in terms of the mood eigenstates) is equal to state $|light\rangle$.

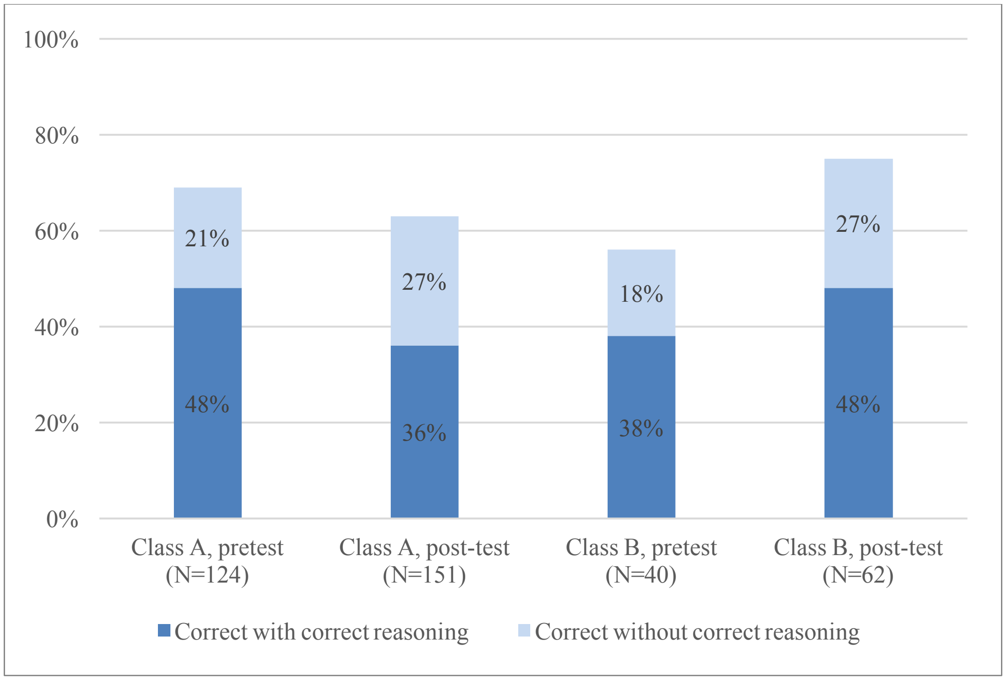


Figure 5.3. Student performance on the pretest (Task 4.2) and the post-test (Task 5.3, question 2)

Figure 5.3 shows the percentage of correct responses for class A and class B on both pretest and post-test (the second question of Task 5.3). The dark shading in the figure represents the percentage of correct responses with correct reasoning, and the light shading represents the percentage of correct responses without correct reasoning.

In class A, student performance on the post-test is slightly lower than pretest. For the prevalence of correct responses, the difference is not statistically significant based on the results from both Fisher's exact test (p = 0.370) and McNemar's test with matched pre- and post-test data (p = 0.877). However, for the prevalence of correct responses with correct reasoning, the results from the two statistical tests are not consistent. The result from Fisher's exact test (p = 0.066) suggests the difference is not statistically significant but the result from McNemar's test (p = 0.030) suggests the difference is statistically significant.

In class B, student performance on the post-test is slightly better than pretest. For the prevalence of correct responses, the result from Fisher's exact test (p = 0.039) suggests that the difference is statistically significant, while the result from McNemar's test (p = 0.080) suggests the difference is not statistically significant. For the prevalence of correct responses with correct reasoning, the results from both tests suggest that the difference is not statistically significant (p = 0.313 from Fisher's exact test and p = 0.096 from McNemar's test).

The results from both classes seem to suggest that student performance on the post-test (mouse context) is not better than pretest (spin context). However, it is not clear whether the prevalence of correct answer with correct reasoning on the post-test is (statistically) significantly lower than pretest since the results from the two classes do not seem to be consistent.

Recall that to answer the post-test (the second question in Task 5.3) correctly, students need to recognize that state $|\phi\rangle$ is the light eigenstate, and state $|\chi\rangle$ is a superposition of weight eigenstates. Since the first question in Task 5.3 (see Task 2.6 in Figure 2.5 in chapter 2) asks for the inner product between a "happy" and "light" mouse, examining student responses to the first question may provide some insights into the lack of positive results on the post-test. As discussed in section 2.4.1, only about half of the students seemed to recognize state $|\phi\rangle$ is an eigenstate weight. These students either wrote $|\phi\rangle = |light\rangle$ or $\langle\phi|light\rangle = 1$. We speculate that the hypothetical mouse context is abstract and thus students have difficulty in making inference of quantum states from experimental results. The linear relationships between the mood eigenstates and the weight eigenstates are required to recognize the measurable effects of the relative phase. Task 5.3 requires students to determine the linear relationships rather than using them directly, which is the case for the spin context. It seems that determining the linear relationships between different bases in the mouse context is challenging for students.

The results suggest that the hypothetical mouse task requires a large step of transfer of what students are shown in the tutorial. The expectation of students being able to correctly answer the mouse question may not be realistic. Therefore, we designed other tasks that require a smaller step of transfer discussed below.

5.2.3.b Tasks with continuous cases

We used Task 4.3 given previously (see Figure 4.2), and two new tasks, Task 5.4, and Task 5.5 (see Figure 5.4) as a second assessment of the effectiveness of the modified tutorial *Superposition in quantum mechanics*. Task 4.3 served as a tutorial pretest, which was given after lecture instruction on the infinite square well. Task 5.4 and Task 5.5 were two post-tests. Task 5.4 was administered on the final exam in class B (Summer 2017), and Task 5.5 was administered on the final exam in class C (Winter 2018).

Recall that in Task 4.3, students consider two ensembles of particles in the infinite square wells. The algebraic expressions of the position-space wave functions for the particles in the two ensembles are given. They wave functions differ only by a relative phase between the energy eigenfunctions. Students are asked whether or not there is an experiment that can be used to distinguish the ensembles. Similar to Task 4.3, the position-space wave functions in Task 5.4 and Task 5.5 also differ only by a relative phase between the energy eigenfunctions and students were asked whether the particles are experimentally distinguishable. However, there are two major differences: (1) the potential wells are unknown, and (2) the energy eigenfunctions are presented graphically.

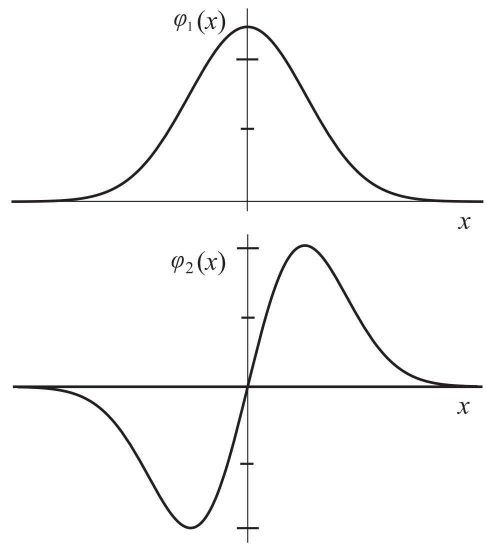
Task 5.4

Consider two superposition ensembles A and B, of identical particles each in an unknown potential. The wave functions of particles in ensembles A and B at $t = 0$ are given below.

$\psi_A(x) = 1/\sqrt{2}(\varphi_1(x) + \varphi_2(x))$ $\quad \psi_B(x) = 1/\sqrt{2}(\varphi_1(x) - i\varphi_2(x))$

Note that $\varphi_1(x)$ and $\varphi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) eigenfunctions with energy eigenvalues E_1 and E_2, respectively. $\varphi_1(x)$ and $\varphi_2(x)$ are both entirely real as shown at right.

Is there an experiment that can be used at $t = 0$ to distinguish the particles in ensemble A from the particles in ensemble B? Explain your reasoning.



Task 5.5

Consider two superposition ensembles A and B, of identical particles each in an unknown potential. The wave functions of particles in ensembles A and B at $t = 0$ are given below.

$\psi_A(x) = 1/\sqrt{2}(\psi_2(x) - \psi_1(x))$ $\quad \psi_B(x) = 1/\sqrt{2}(\psi_2(x) - i\psi_1(x))$

Note that $\psi_1(x)$ and $\psi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) eigenfunctions with energy eigenvalues E_1 and E_2, respectively. $\psi_1(x)$ and $\psi_2(x)$ are both entirely real as shown at right.

Is there an experiment that can be used at $t = 0$ to distinguish the particles in ensemble A from the particles in ensemble B? Explain your reasoning.

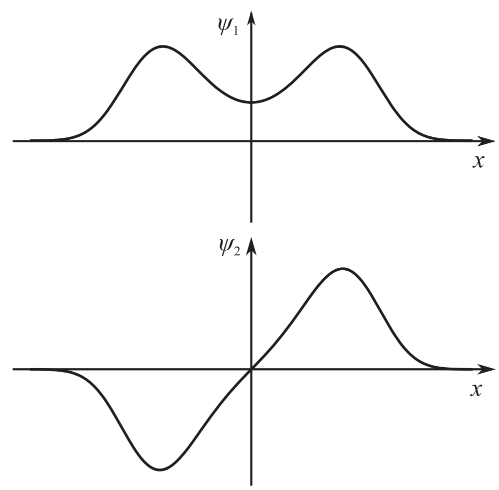


Figure 5.4. Tasks involving unknown potential wells given on final exams in PHYS 225

On both Task 5.4 and Task 5.5, the particles can be distinguished by measuring position. To make a comparison of position-space probability density, one can focus on a specific region. For instance, one can compare the probability density on the right half of the well. For wave functions that have purely real values (e.g., $\psi_A(x) = \frac{1}{\sqrt{2}}(\psi_2(x) - \psi_1(x))$, one can sum or subtract the values in the right half before taking the modulus squared. For wave functions that have both real and imaginary parts (e.g., $\psi_B(x) = \frac{1}{\sqrt{2}}(\psi_2(x) - i\psi_1(x))$, one needs to take the modulus squared of each part before doing the sum (since the cross terms cancel). Therefore, the probability of finding the particle on the right half is greater for ensemble A than ensemble B on Task 5.4; the probability of finding the particle on the right half is less for ensemble A than ensemble B on Task 5.5.

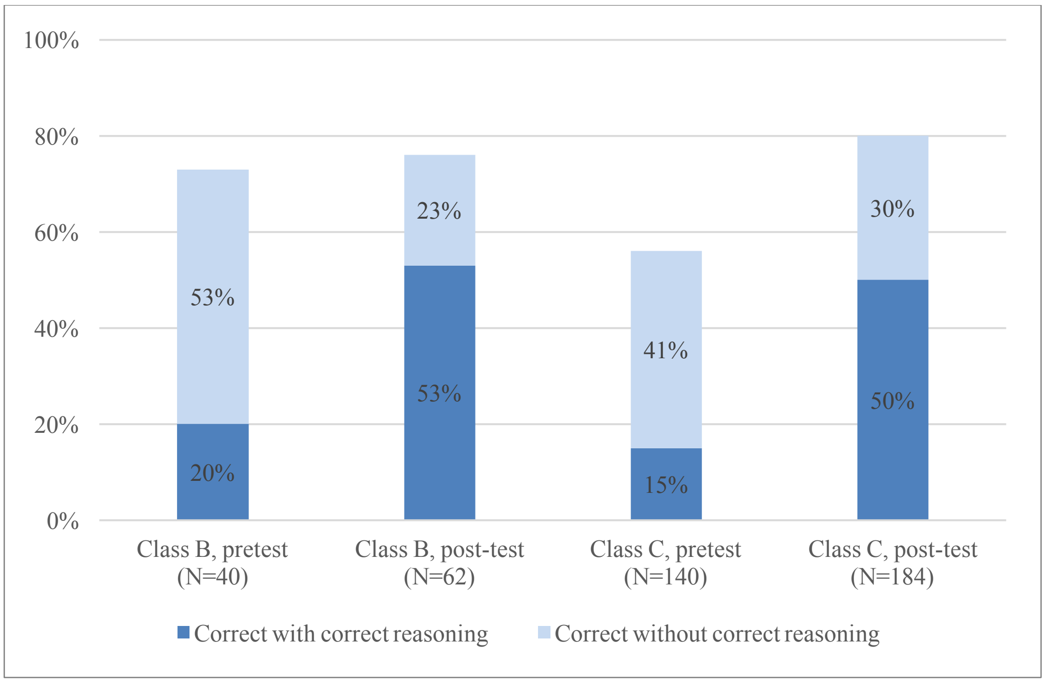


Figure 5.5. Student performance on the pretest (Task 4.3) and the post-tests (Task 5.4 and Task 5.5).

Figure 5.5 shows the percentage of correct responses for class B and class C on both pretest and post-test. The dark shading in the figure represents the percentage of correct responses with correct reasoning. The light shading represents the percentage of correct responses without correct reasoning. For both the pretest and the post-tests, we consider responses to be correct with correct reasoning if the students identified the correct observable (e.g., position or momentum) that can be measured to distinguish the particles or if students pointed out that the probability densities are different. Detailed description of how probability densities are different was not required.

For class B, the percentage of correct responses on pretest is very similar to the post-test. (Note that the percentage of correct response on the pretest in class B is much higher than the aggregated results (over different classes). See chapter 4, section 4.4. This may be due to a random fluctuation since the number of students in class B is small.) However, the percentage of correct responses with correct reasoning on the post-test is much greater than that on the pre-test. We used both Fisher's exact test and McNemar's test (with matched data on pre- and post-tests) to evaluate the improvement. The results from both tests suggest that the improvement in correct response with correct reasoning in class B is statistically significant ($p < 0.001$ from Fisher's exact test and $p < 0.001$ from McNemar's test).

For class C, both the percentage of correct responses and the percentage of correct responses with correct reasoning were greater on the post-test. The difference in percentages of correct response is statistically significant ($p < 0.001$ from both Fisher's exact test and McNemar's test). The difference in percentages of correct response with correct reasoning is statistically significantly as well ($p < 0.001$ from both Fisher's exact test and McNemar's test).

In summary, the results suggest that the modified tutorial *Superposition in quantum mechanics* appears to improve student ability to recognize the measurable effect of relative phase in cases that involve position-space wave function. After tutorial instruction, a significantly greater portion of students correctly explained how states differ only by a relative phase can be experimentally distinguishable. However, there still appears to be room for improvement. Some students graphed probability density incorrectly, or wrote incorrect expressions for probability density. The results suggest that after tutorial instruction, some students still had difficulty with determining the modulus squares of complex-valued functions or with graphing the modulus squares of complex-valued functions.

5.3 Modification of Tutorial *Time Dependence in Quantum Mechanics*

It has been our experience that students do not develop conceptual understanding and reasoning skills at once but rather at stages, especially when the concepts are challenging. The original version of the tutorial *Time dependence in quantum mechanics* does not directly address student difficulty in relating relative phase and probability. Therefore, we modified the tutorial such that students are provided opportunities to revisit the concept relative phase. The original and the modified versions are in Appendix A, Figure A.15 and Figure A.16.

In the original version of the tutorial *Time dependence in quantum mechanics,* students consider the time dependence of wave functions and their associated probability densities. They are guided to represent both the real and imaginary parts of a wave function. In particular, they are provided a visualization tool that allows them to represent a (complex) wave function in a three-dimensional space. The tool consists of a set of real and imaginary axes printed on transparencies with a third axis representing a one-dimensional position space. Students use this tool to first consider the time dependence of two energy eigenstates and their associated probability densities. They then repeat this exercise for a superposition of energy eigenstates. They are guided to use the transparencies to visualize each energy eigenstate at a given time.

Although the original tutorial is intended to help students recognize that each (energy) basis state (in the superposition state) rotate on a complex plane with a unique rate, it does not explicitly guide students to relate the relative phase between the energy eigenstates to the time dependence of probability density. Therefore, we modified the tutorial such that it makes use of phasors to facilitate students in visualizing the relative phase. Note that the phasors representation is also used in the tutorial *Two-state time dependence* (see Appendix A, Figure A.17) intended for sophomore-level students. Integrating phasors in the tutorial *Time dependence in quantum* *mechanics* is intended to reinforce student understanding of phase and student fluency with phasor representation.

In what follows, we describe several major modifications to the tutorial. First, students are guided to represent the phase of each energy eigenstate at several specific times on the complex plane. In particular, they are prompted to sketch a phasor as an arrow such that the angle between the phasor and the positive real axis equals the phase of the energy eigenstate. Students are then prompted to think about how the relative phase between the energy eigenstates change when the entire superposition state evolves in time. Students are expected to recognize that the position-space probability density changes in time due to that the relative phase between the energy eigenstates changes in time.

We also added a new exercise at the end to help student generalize the idea that position-space probability density depends on the relative phase between the energy eigenstates. Students consider the state at an unknown time, but the relative phase is the same as the initial state (at $t = 0$). They consider whether or not the probability density at that time is the same as the probability density for the initial state (at $t = 0$). They are also guided to consider how the probability density depends on the relative phase as the state evolves in time.

We are planning to assess the new version of tutorial in the future. In particular, we are planning to probe student ability to use phasor representation in detail.

5.4 Summary

The results discussed in the chapter 4 suggest that many students do not recognize that relative phase can affect probabilities. Proficiency with determining modulus squared of complex numbers do not seem to be the primary barrier. Rather, students appear to believe that the relative phase does not affect probabilities or modulus squared. These findings have driven the development of three tutorials intended to help students develop a functional understanding of relative phase. The tutorials *Quantum interference with spin states* and *Superposition in quantum mechanics* have been implemented in the sophomore-level course. The tutorial *Time dependence in quantum mechanics* has been used in the junior-level course.

The development of the tutorial *Quantum interference with spin states* illustrates how curriculum development can, in turn, drive a further investigation into student understanding. The tutorial *Quantum interference with spin states* makes use of a scientific inquiry approach. Students are guided to infer the complex coefficients of quantum states from statistical experimental outcomes. The results suggest that the ideas developed in the tutorial *Quantum interference with spin states* are challenging for many sophomore-level students. We conducted a follow-up study and found that many students do not spontaneously use complex numbers for coefficients. When asked whether there is more than one possible state that can describe the experimental results (the results for measuring one observable), many do not recognize that there are many possible solutions with coefficients that can have different phases. Many seem to believe that there is only one possible state given a particular set of statistical results. Others tend to answer that there is more than one possible answer, but argue incorrectly that this is the case since one can express the state in different bases. The results from the further investigation provide insights into potential instructional strategies to improve the tutorial. We are planning to modify and assess the tutorial in the future.

We have also revised the tutorial *Superposition in quantum mechanics.* Our preliminary assessment results suggest that the revised version of the tutorial appears to be effective at improving student ability to recognize the measurable effect of relative phase. After tutorial instruction, many students recognized that position-space wave functions that differ only by a relative phase between the energy eigenfunctions can be distinguished by position measurements. Despite the promising results from assessment, there still appears to be room for improvement. Some students had difficulty with determining computing modulus squared of complex-valued wave functions, or with graphing probability density even after tutorial instruction. Further modifications to the tutorial that target on these specific difficulties are necessary.

This chapter also discusses several modifications to the tutorial *Time dependence in quantum mechanics* used in the junior-level course. This is the first of a sequence of three tutorials intended to facilitate students to reason about time dependence and measurements. The modified version of the tutorial guides students to make use of phasors representation. It is intended to make clear to students that the position-space probability density changes in time is due to that the relative phase between the energy eigenstates changes in time. We are planning to assess the modified version of the tutorial in the future.

Chapter 6. Student Ability to Apply Superposition to Interference of Classical Waves

The results presented in the previous two chapters suggest that many students do not recognize the relationship between relative phase and quantum interference. This finding prompted us to ask whether students recognize the analogous relationship between phase difference and classical wave interference. Since students are typically taught classical interference in introductory-level physics courses before they take quantum mechanics, we decided that investigating introductory students' ability to apply superposition to wave interference might provide insights into quantum students' understanding of relative phase in superposition states.

In order to understand interference phenomena of classical waves in the context of water or light, students need to be able to reason about superposition of two or more waves. They need to be able to do point-by-point addition of waves, and to recognize what does and does not change at a particular point in space as time passes. The ways in which the waves combine depend on the phase difference.

Physics education literature has documented results from studies on student understanding of wave mechanics from various perspectives [38,40,73-76]. However, research that specifically focuses on student understanding of superposition and phase difference is less extensive. In this chapter, we discuss an investigation into student functional understanding of phase difference. In particular, we probe the extent to which students are able to use phase difference to reason about interference of classical waves. The specific research questions are discussed in section 6.2.

As discussed in chapter 1, the introductory-level wave and optics course at the UW makes use a sequence of four tutorials on interference developed by UW PEG. The sequence starts with *Two-source interference* in the context of water. The second of the sequence is *Wave properties of*

Light, which guides students to recognize the analogy between water waves and light waves. The tutorials *Multiple-slit interference* and *A model for single slit diffraction* are a continuation of the first two tutorials on interference. During the tutorial *Multiple-slit interference*, students are provided transparencies of sinusoidal waves that they can overlay on each other to help them visualize superposition of more than two waves. The concepts of path length difference and phase difference are key elements in the models of the entire sequence of tutorials. Note that phasors were not included in this sequence prior to the research introduced in this chapter.

This chapter starts with a literature review of relevant research on student understanding of superposition of waves in section 6.1. We then discuss the research goals in section 6.2. In sections 6.3 and 6.4, we discuss results from our investigation into student ability to differentiate between phase difference and path length difference and student ability to apply superposition in multiple-slit interference. This chapter ends with a summary in section 6.5.

6.1 Prior Research

In this section, we first briefly review prior research from the physics education research community on student understanding of wave optics. We then narrow the focus to relevant studies conducted by UW PEG.

The interference and diffraction of light illustrates the wave behavior of light. However, in working with university students, Maurines found that many tend to use a geometrical model to reason in situations where wave optics is appropriate [75]. In particular, even after instruction, many students tend to use light rays rather than phase and the Huygens-Fresnel principle to reason about diffraction.

Sengoren found that students often do not realize that there is a continuity of brightness in interference patterns [76]. Rather they think of an interference pattern (e.g., a two-slit pattern) consists of alternate bands of uniform bright and dark.

Of particular relevant to the study discussed in this and the next chapter, Mesic et al. compared three different instructional approaches to helping high school students to visualize light waves [74]. In these three approaches, light waves are represented as sinusoidal curves, oscillating electric field vectors, or phasors. The student participants were randomized into three groups, each received one of the instructional approaches. After instruction, students completed a survey of conceptual questions on interference and diffraction. Students who were taught phasors or oscillating electric field vectors outperformed students who were taught the sinusoidal curve representation. The results suggest that visualizing light waves with phasors or oscillating electric field vectors is a more effective approach.

The study discussed in this chapter directly builds on the research conducted by former members of UW PEG, Bradley Ambrose and Karen Wosilait. Ambrose et al. investigated student understanding of single-slit diffraction and double-slit interference after traditional lecture instruction [38]. They identified several common and persistent difficulties students encounter. For example, many students apply the model from geometrical optics when physical optics is appropriate and vice versa. It is also common for students to combine elements from both into a "hybrid" model. For example, some students seem to believe that when one of the two slits is covered, half of the bright fringes of the interference pattern would disappear.

Ambrose et al. also found that many students incorrectly apply key aspects of the wave model when reasoning about superposition qualitatively. For example, many students fail to use path length (or phase) difference to infer about interference pattern. Instead, students sometimes use path length to decide whether waves interfere constructively or destructively. For instance, students seem to believe that the path length difference becomes negligible at large distances.

Wosilait conducted further investigations into student ability to apply a wave model for interference and diffraction of light. Her dissertation [77] documented in great detail the development of a sequence of four tutorials on interference driven by the results from her investigation (*Two-source interference, Wave properties of light, Multiple-slit interference,* and *A model for single-slit diffraction*). Wosilait et al. showed that this sequence of tutorials can be effective at addressing several difficulties that students encounter while applying a wave model to interference and diffraction [40]. In particular, the tutorials are successful at supporting students in relating path length difference to interference.

In summary, studies on student understanding of interference and diffraction involve a variety of aspects. Many focus on the model (a wave model or a model from geometrical optics) students use to reason about interference. Very few explicitly probes the role that phase difference plays in student reasoning. It is not clear from the literature that whether or not students recognize the similarities and differences between phase difference and path length difference. Prior research also does not demonstrate to what extent students are able to apply these concepts in qualitative questions.

6.2 Overview and Goals

In this chapter, we describe a study into student functional understanding of phase difference of classical waves. This study probes, in depth, how students use phase difference to reason about superposition. In particular, we intend to answer the following questions:

1. To what extent are students able to determine phase difference based on an interference pattern?
2. To what extent do students differentiate between phase difference and path length difference?
3. To what extent are students able to apply superposition to multiple-slit interference?

Section 6.3 is intended to answer the first two research questions. The third question is answered in section 6.4. We examine in detail how students approach qualitative questions on multiple-slit interference. We also identify the specific difficulties students encounter when reasoning about superposition. Lastly, we summarize our research findings and discuss the instructional implications in section 6.5.

6.3 Student Ability to Differentiate Between Phase Difference and Path Length Difference

In this section, we discuss results from our study on student ability to differentiate between phase difference and path length difference. There were two stages in our investigation. Section 6.3.1 presents stage one, in which we examined how students use interference pattern to determine phase difference. The results suggest that many students seem to confuse phase difference with path length difference. This finding motivated the investigation in stage two, as will be discussed in section 6.3.2, which was based on a task we used that prompted students to determine path length difference and source separation. This task allowed us to gain insights into whether or not student recognize the difference between path length difference and phase differences.

6.3.1 Stage one: Examining student ability to determine phase difference from an interference pattern

In section 6.3.1.a, we discuss a task that was used to examine student ability to determine phase difference from an interference pattern. Section 6.3.2.b presents the overall student performance. The specific student difficulties are discussed in section 6.3.2.c.

6.3.1.a Task design

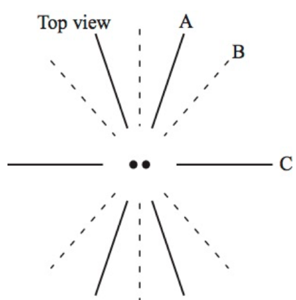
Figure 6.1 shows a task that was administered on a midterm exam in the introductory wave and optics course (PHYS 123). Students had received relevant lecture instruction and completed tutorials *Two-source interference, Wave properties of light,* and *Multiple-slit interference.*

In Task 6.1, students consider two point sources that are producing periodic waves in a ripple tank of water. They are given the interference pattern with all the nodal lines and lines of maximum constructive interference. In part A, students are asked whether or not the sources are in phase. In part B, students are prompted to determine the phase difference for line A and line B, respectively.

It is worth noting that students had seen such interference pattern of water waves in the tutorial. Besides, students had experience determining the path length difference and phase difference. However, the cases in the tutorial involve sources that are in phase, while the sources in Task 6.1 are out of phase. We chose a case in which the sources are out of phase because it required students to apply the concepts they had studied in new situation. This new context allowed us to examine to the extent to which students are able to relate interference patterns to phase difference.

Task 6.1

Two point sources, S_1 and S_2, produce periodic waves of wavelength λ in a ripple tank of water. All of the nodal lines (dashed) and lines of maximum constructive interference (solid) produced by the two sources are as shown at right.



A. Are the two sources S_1 and S_2 in phase? Explain. If there is not enough information to determine, state so explicitly.
B. Determine the difference in phase $\Delta\varphi$ of the waves that reach each point along line A and each point along line B. Briefly explain.

Figure 6.1. A task involving phase difference given on a midterm exam in PHYS 123.

To decide whether or not the sources are in phase in part A, one can make use of the vertical line between the sources. Since every point along the vertical line is equidistant from both sources, the path length difference for every point along this line is zero. The vertical line in this case is a nodal line, meaning that waves travelling the same distance from each source are destructively interfering. This can only occur if one source is creating a crest at the same time as the other is creating a trough. Therefore, the sources are 180^{o} or π out of phase. Note that students were not required to determine the phase difference between the sources. For part B, we are told that line A is a line of maximum constructive interference and line B is a nodal line. Thus, the two waves are in phase (zero or a multiple of 2π phase difference) at each point along line A and the two waves are completely out of phase (with a phase difference of π or an odd integer of π) at each point along line B.

6.3.1.b Overall student performance

Table 6.36 shows the percentage of correct answer for each question in Task 6.1. In general, students performed much better on part A than on part B.

On part A, most (about 80%) of the students correctly answered that the sources are out of phase. We examined the explanations students provided and found that most of the students seemed to arrive at their answers based on the fact that the vertical line is a nodal line. The results suggest that students are able to recognize whether the sources are in phase or out of phase from the interference pattern.

On part B, only about 30% of the students correctly determined the phase difference along line A. These students either answered that the phase difference is zero or 2π. For line B, only about 18% of the students correctly determined the phase difference.

Table 6.36. Percentage of students giving correct asnwers on Task 6.1

	Part A	Part B	
	Sources	$\Delta\varphi_A$	$\Delta\varphi_B$
Correct answer (N = 185[a])	80%	30%	18%

[a]PHYS 123A[3]: SPR 16.

The common incorrect answers to each question in part B is shown in Table 6.37. On each question, about half of the students gave answers that were written in terms of wavelength λ. For line A, the common incorrect answers involve λ and $\lambda/2$. For line B, the common incorrect answers include $3\lambda/2$, λ and $\lambda/2$.

Table 6.37. Common incorrect answers to part B of Task 6.1

$\Delta\varphi_A$		$\Delta\varphi_B$	
λ (or mλ)	29%	$3\lambda/2$ (or $\lambda/2$)	36%
$\lambda/2$	21%	λ	18%
π	9%	2π	5%

6.3.1.c Identification of student difficulties

The results above show that many students incorrectly determined the phase differences for the nodal lines and lines of maximum constructive interference. In this section, we identify the specific difficulties students encountered when they determined the phase differences. These

[3] The letter represents the lecture section.

difficulties are not necessarily mutually exclusive. Therefore, we do not report the prevalence for each difficulty.

6.3.1.c.1 Misuse of memorized rules

We found that many students used rules that hold true only for cases in which the sources are in phase. The rules students commonly used include:

(1) Treating lines of maximum constructive interference as having path length differences of integer wavelength, and nodal lines as having path length differences of half integer wavelength.

(2) Treating a path length difference of λ as corresponding to a phase difference of 2π.

Students who used these rules tended to have determined that $\Delta\varphi_A = \pi$ and $\Delta\varphi_B = 2\pi$. For instance, a student wrote *"Knowing that φ [phase difference] is equal to the path difference* $\times$ *[multiplied by]* $\frac{2\pi}{\lambda}$*, we can calculate the phase.* $\Delta\varphi_A = \frac{\lambda}{2}\times\frac{2\pi}{\lambda} = \pi$. $\Delta\varphi_B = \lambda\times\frac{2\pi}{\lambda} = 2\pi$*."* This student determined the path length differences first. Then the student converted the path length differences to phase differences using a formula that is only appropriate when the sources are in phase.

6.3.1.c.2 Tendency to conflate the concepts of phase difference and path length difference

We found that it is very common that students incorrectly expressed phase difference in terms of λ. These students did not seem to recognize that phase (or phase difference) does not have units. The explanations students provided seemed to suggest that many students, to some extent, conflated the concepts of phase difference and path length difference, which is typically written in terms of wavelength. We will use two example student responses to illustrate this interpretation of student difficulty.

One of the students answered, *"In order to construct a maximum constructive interference, the difference in phase* $\Delta\varphi$ *must be a multiple of* λ*. A is the first solid line so* $\Delta\varphi_A = \lambda$*. For a nodal line,* $\Delta\varphi$ *has to be* $(m + 0.5)\lambda$ *where* m *is an integer and since B is right after A,* $\Delta\varphi_B = 1.5\lambda$*."* This student does not seem to differentiate between path length difference and phase difference. It is worthwhile to point out that the answers would have been correct if λ does not represent one wavelength, but rather represents the angle after one period (the period of a sine function, which is usually noted by 2π or 360°). If the student this simply confuse the concept of phase difference with path length difference, we would expect the student to answer that $\Delta\varphi_A = \lambda/2$ and $\Delta\varphi_B = \lambda$ rather than that $\Delta\varphi_A = \lambda$ and $\Delta\varphi_B = 1.5\lambda$. This student seems to conflate, to some extent, the concepts of phase difference and path length difference.

Another student who arrived at the same answers ($\Delta\varphi_A = \lambda$ and $\Delta\varphi_B = 3\lambda/2$) explained that: *"Following the pattern set by the* ΔD *(difference in path length), each line should be* $\lambda/2$ *from the last. The point sources are also out of phase though by a factor of* $(m + 1/2)\lambda$ *(since the* $\Delta D = 0$ *line is a nodal line). I will assume* $\left(m + \frac{1}{2}\right)\lambda = \lambda/2$*."* (Note that on the diagram, the student labelled line A "$\Delta D = \lambda/2$" and line B "$\Delta D = \lambda$".) This student first determines *correctly* the path length differences. To determine the phase differences, the student adds the phase difference (expressed in terms of λ) between the sources to the path length differences. If instead, the student converts the path length difference of $\lambda/2$ into a phase difference of π, then adds it to the phase difference between the sources, the student would arrive at a correct answer. This student seems to use elements from both concepts: (1) path length difference written in terms of wavelength, and (2) phase differences can be added. This suggests that this student does not simply confuse phase difference with path length difference, but rather conflate these two concepts.

The results seem to suggest that some students have a “hybrid” model for phase difference. They seem to treat path length difference as phase difference and express phase difference in terms of wavelength. These students do not seem to treat wavelength as a quantity of length, but rather as the phase for one period (a period of a sine function, not to be confused with a quantity of time). This may be due in part to the fact that phase is an abstract concept for the students and they tend to make sense of this abstract concept based on a more concrete concept, wavelength.

6.3.2 Stage two: Examining student functional understanding of path length difference

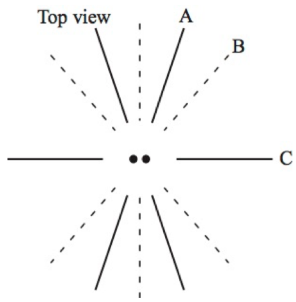
The results from last section show that when asked to determine phase difference, students tend to conflate path length difference and phase difference. This motivated our further investigation. In order to test whether this difficulty arises only when students are prompted to determine phase difference, we designed another task that asked students to determine path length difference and source separation. In this section, we first describe the task that was used to probe student functional understanding of path length difference in 6.3.2.a. We then discuss student performance and analysis in 6.3.2.b.

6.3.2.a Task design

Task 6.2 was administered on final exams in PHYS 123, as shown in Figure 6.2. Similar to Task 6.1, Task 6.2 involves two sources of water waves that are out of phase. However, in this case, students are told that the sources are 180^{o} out of phase (rather than having to infer this information from the diagram). They are asked to determine the path length difference for each line and source separation. Having prompts that were different from the one in Task 6.1 allows us to test whether the tendency to conflate path length difference and phase difference arises only when a specific prompt is used.

Task 6.2

Two point sources, S_1 and S_2, produce periodic waves of wavelength λ in a ripple tank of water. The sources tap the water 180° out of phase. All of the nodal lines (dashed) and lines of maximum constructive interference (solid) produced by the two sources are as shown.



1. Which of the following choices best represents the path length differences ΔD_A, ΔD_B, and ΔD_c corresponding to each of the lines A, B, and C?
   A. $\Delta D_A = \lambda/2$; $\Delta D_B = \lambda$; $\Delta D_C = \lambda/2$
   B. $\Delta D_A = \lambda/2$; $\Delta D_B = \lambda$; $\Delta D_C = 3\lambda/2$
   C. $\Delta D_A = \lambda$; $\Delta D_B = \lambda/2$; $\Delta D_C = \lambda$
   D. $\Delta D_A = \lambda$; $\Delta D_B = 3\lambda/2$; $\Delta D_C = 2\lambda$

2. What is the source separation d of the two sources S_1 and S_2 in terms of λ?
   A. $d = \lambda/2$
   B. $d = \lambda$
   C. $d = 3\lambda/2$
   D. $d = 2\lambda$

Figure 6.2. A task involving path length difference given on final exams in PHYS 123.

The questions were in multiple-choice format. The answer options were designed based on student responses to Task 6.1. To answer question 1, students need to recognize that any point along the vertical line is equal distance from the sources. Since line A is the first line of maximum constructive interference next to the vertical line and since the sources are 180° out of phase, the path length difference for line A is $\lambda/2$. Similarly, the path length difference for line B is λ and for line C is $3\lambda/2$. Thus, the correct answer to question 1 is answer B. For question 2, students need to recognize that source separation is equal to the path length difference for the horizontal line. Since the path length difference for the horizontal line (line C) is $3\lambda/2$, the source separation is $3\lambda/2$.

The incorrect answers in both questions 1 and 2 were used to elicit the error of not differentiating between path length difference and phase difference. We hypothesized that students who conflate path length difference and phase difference would add the phase difference between the sources to the path length difference for line C, $3\lambda/2$, and thus conclude that 2λ is the answer for line C. Those students would then answer that the source separation is 2λ. As a result, those students would choose answer D in both questions. The other answers (in either question 1 or 2) were intended for students who think that a path length difference of $3\lambda/2$ is the same as $\lambda/2$ (and a path length difference of 2λ is the same as λ). This type of reasoning would also demonstrate not differentiating between path length difference and phase difference since a phase difference of 3π is equivalent to π.

6.3.2.b Student performance and analysis

On part 1 of Task 6.2, the path length difference question, about 58% of the students chose the correct answer. Only a few students chose either answer A or C. However, about 38% chose answer D. It seemed that many students did not differentiate between path length difference and phase difference when they were asked to determine path length difference.

About half (54%) of the students answered correctly on part 2, the source separation question. This is similar to the percentage of correct answer on the path length difference question. It seems that about half of the students did not differentiate between phase difference and path length difference when determined the sources separation. Besides, about 22% (of all students) chose answer D, 16% chose answer A, and 8% chose answer B. We speculate that students who chose answer D determined the path length difference for line C incorrectly, but were being consistent on both questions. Students who chose A (or B) were likely to believe that a path length difference of $3\lambda/2$ is equivalent to $\lambda/2$ (or a path length difference of 2λ is equivalent to λ).

Table 6.38. Student performance on *Path length difference* and *Source separation* questions in Task 6.2

$N = 395$[a]		*Source separation* question	
		Correct (Answer C)	Conflating path length difference and phase difference (the other answers)
Path length difference question	Correct (Answer B)	42%	16%
	Conflating path length difference and phase difference (the other answers)	11%	30%

[a]PHYS 123B: WIN13. PHYS 123C: SPR13.

To figure out whether or not students are being consistent on both questions, we considered individuals' answers to both questions. The prevalence is shown in Table 6.38. About 42% answered both questions correctly, and about 30% made the error of conflating path length difference and phase difference on both questions. However, about 27% are not being consistent. The results seem to show that some students consistently made the same error in different questions, and some made the error only in one of the questions.

6.3.3 Summary

This section discusses student ability to differentiate between phase difference and path length difference. We used tasks in which the sources are out of phase, and prompted students to determine phase difference, path length difference, and source separation. With all three prompts, we have evidence showing that many students do not differentiate between phase difference and path length difference. Many students tend to express phase difference in terms of wavelength, and to add the phase difference between the sources to both path length difference and source separation. These students do not seem to treat path length difference or source separation as length quantities. This suggest that the confusion between phase difference and path length difference is not simply a confusion about the terminology, but demonstrate student conceptual difficulties.

6.4 Student Ability to Apply Superposition to Multiple-slit Interference

This section presents results from our investigation into student ability to apply superposition to multiple-slit interference. In particular, we examine (1) the approaches students use to reason about multiple-slit interference and (2) the extent to which students are able to use path length difference and/or phase difference to reason about superposition.

This section starts with 6.4.1, in which we describe the tasks that were used to examine student reasoning. We then discuss overall performance in section 6.4.2. Specific difficulties students encounter when reasoning about superposition are discussed in section 6.4.3.

6.4.1 Task design

To probe student thinking about superposition, we designed two tasks (Task 6.3 and Task 6.4), as shown in Figure 6.3. The tasks were administered on midterm exams in PHYS 123. Students were taught interference and diffraction in lecture, and had completed the entire sequence of tutorials on interference (including *Two-source interference, Wave properties of light, Multiple-slit interference,* and *A model for single-slit diffraction*).

Task 6.3: On Task 6.3, students consider an interference pattern on a distant screen due to two slits that are separated by a distance d. Point A marks a minimum and point B marks a maximum on the screen. Students are then told that two additional slits are added, such that all four slits are now separated by equal distances $d/2$. Students are asked, with four slits, whether point A would be a principal maximum, a minimum, or neither. They are then prompted to make a similar prediction for point B.

One way to answer these questions is to consider the path length difference between adjacent slits. With the original two slits, the path length difference corresponding to point A is $5\lambda/2$ and the path length difference corresponding to point B is λ. With four slits, the path length difference between adjacent sits are $5\lambda/4$ for point A and $\lambda/2$ for point B. For four slits, path length differences from adjacent slits to points of minimum intensity are integer multiples of $\lambda/4$ that are not whole number multiples of λ (i.e., $\lambda/4$, $2\lambda/4$, $3\lambda/4$, etc.). Therefore, both point A and point B would become minima. Students could also reason alternatively that, for point A, light from the original slits with a path length difference of $5\lambda/2$ cancels. That means light from the additional pair of slits (which has the same separation) would also cancel at point A due to the same path length difference.

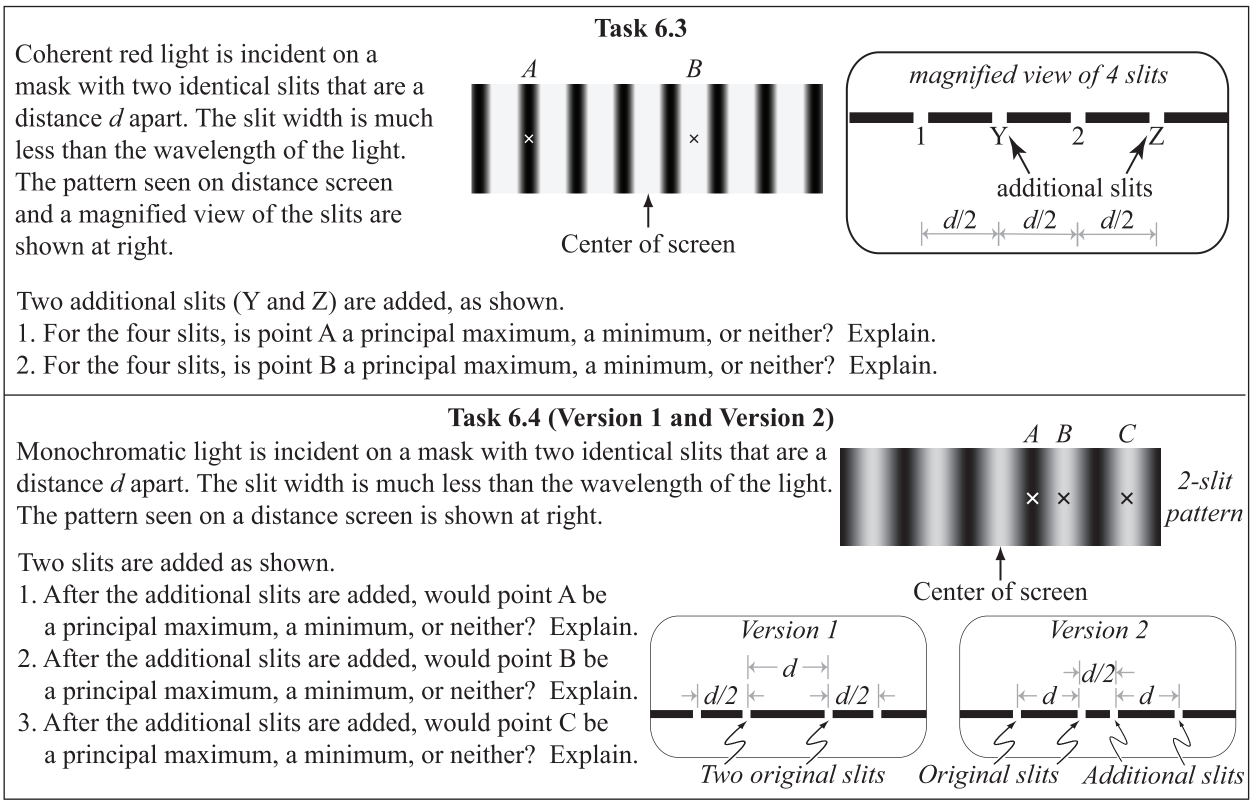


Figure 6.3. Tasks involving multiple-slit interference given on midterm exams in PHYS 123.

Task 6.4: There are two different versions of Task 6.4. In both versions, students first consider an interference pattern due to two slits and three points A, B, and C on the screen. They are told that two additional slits are added. In both cases, the slits are no longer equally spaced. The only difference between the two versions is that the additional slits are added in different ways, as shown in Figure 6.3. In both versions, students are asked to consider for each point, whether it would be a principal maximum, a minimum, or neither.

To answer the questions in Task 6.4, one can no longer use the rules for slits that are equally-spaced, but one can still use the path length difference for pairs of slits to reason about superposition. With the original two slits, point A is the first minimum, point B is the first principal maximum, and point C is the second principal maximum. Thus, the path length difference for point A, point B, and point C is $\lambda/2$, λ, and 2λ, respectively.

Version 1 (V1): On V1, the path length difference from the additional slits to point A is λ. Therefore, light from the additional slits constructively interference. However, since light from the original slits cancel, the result is neither a principal maximum or a point of zero intensity at point A. For point B, one can use different pairs of slits. The path length difference from the left two slits is $\lambda/2$. Therefore, light from the left two slits cancel. Light from the right two slits also cancel for the same reason. Thus, point B would become a point of zero intensity. For point C, the path length difference from the left two slits (or the right two slits) is λ, and the path length difference the original slits is 2λ. Thus, light from all four slits are in phase at point C. Point C would be a principal maximum.

Version 2 (V2): Using similar reasoning that described above, one can determine that on Version 2 (V2), point A would be a point of zero intensity, point B would be a point of zero intensity, and point C would be a principal maximum.

6.4.2 Student performance

Table 6.39 shows the percentage of correct answers given by the students to each question on Task 6.3 and both versions of Task 6.4. (Recall that students had completed all tutorial and lecture instruction on physical optics.) With the two original slits only, on all tasks, point A is a minimum (although not the same minimum), point B is the first principal maximum, and point C is the second principal maximum. To aid in comparing student performance across all tasks, the first row of Table 6.39 shows the different points on the screen and the first column shows the different tasks. This allows for easy identification of patterns across different tasks.

Table 6.39. Percentage of correct answer on each question in Task 6.3 and Task 6.4. The correct is shown in parenthesis.

Points on screen	Point A	Point B	Point C
Task 6.3 ($N = 188$[a])	66% (minimum)	43% (minimum)	N/A
Task 6.4-Version 1 ($N = 177$[b])	62% (neither)	27% (minimum)	73% (neither)
Task 6.4-Version 2 ($N = 167$[c])	35% (minimum)	30% (minimum)	54% (maximum)

[a]PHYS 123A: AUT 09. [b]PHYS 123A: SPR 09. [c]PHYS 123C: SPR 09.

Overall, the percentage of correct answer for each question varies from about 30% to about 70%. On half of the questions, the prevalence of correct answer is lower than 50%. These questions seem challenging for many students. Point B, in particular, seem the most challenging. Across all tasks (no matter the slits are equally-spaced or not), the percentage of correct answer is consistently lower than 50%.

Table 6.40. Percentage of the most common incorrect answer on each question in Task 6.3 and Task 6.4. The most common incorrect is shown in parenthesis.

Points on screen	Point A	Point B	Point C
Task 6.3 (N = 188[a])	22% (neither)	42% (maximum)	N/A
Task 6.4-Version 1 (N = 177[b])	27% (minimum)	50% (maximum)	14% (neither)
Task 6.4-Version 2 (N = 167[c])	54% (neither)	44% (neither)	29% (neither)

[a]PHYS 123A: AUT 09. [b]PHYS 123A: SPR 09. [c]PHYS 123C: SPR 09.

The most common incorrect answer on each question is shown in Table 6.40. For point B, more than 40% chose the most common incorrect answer for each task. On Task 6.3 and Task 6.4-V1, the common incorrect answer for point B is the same. It is likely that students used similar reasoning on both tasks.

6.4.3 Analysis of student reasoning

The low performance by students on some of the tasks above suggest that the tasks were challenging for many students. The answers and reasoning given by students suggest several incorrect patterns of thinking. In particular, the ideas expressed for point B were useful in eliciting student thinking.

Across different tasks, we categorized the approaches students used for point B. These approaches sometimes led to correct answers, and sometimes led to incorrect answers. To illustrate the extent to which students were able to use these approaches productively, we discuss example correct responses. We also characterized and identified the incorrect lines of reasoning students used for each approach. By comparing and contrasting the correct and incorrect responses for each approach, we might gain insights into potential instructional strategies to address student difficulties.

In section 6.4.3.a, we categorize the approaches students used. We then discuss specific difficulties students had in 6.4.3.b.

6.4.3.a Identification of student approaches in answering Tasks 6.3 and 6.4

This section categorizes the approaches students used, which allows us to examine the extent to which students are applying the principle of superposition when they reason about multiple-slit interference. Three major approaches were identified. Note that these categories are not necessarily mutually exclusive. Some students used more than one approach.

6.4.3.a.1 Qualitative discussion of path length difference or phase difference

On all tasks, we found that about 60% of the students used path length difference or phase difference. Relatively few students explicitly determined the phase differences. Most described qualitatively whether the waves are in phase or out of phase. For example, one student on Task 6.4-V1 wrote "*Originally,* $\Delta D_{12} = \lambda$, *but now [slits] 1 and Y is* $\lambda/2$, *out of phase and cancel. The same occurs to [slits] 2 and Z. B is min [minimum].*" This student discussed superposition of light from the two pairs of slits individually and correctly used path length difference to infer that light from each pair are out of phase. The student then correctly stated that point B would become a minimum.

Other students discussed the path length difference between adjacent slits without mentioning phase difference. On Task 6.3, one of the students answered, *"The number of slits double but slit distance halves. B now has* $\Delta D = \lambda/2$. *Thus, it's minimum."* This student used a rule that they discovered from the tutorial that for minima with four slits equally spaced, the path length difference between the adjacent slits need to be a multiple of $\lambda/4$ (but not an integer of λ). Note that this type of answer is much less prevalent on Task 6.4 since the slits are not equally spaced.

6.4.3.a.2 Quantitative discussion of path length difference

On all tasks, about 20% of the students used quantitative reasoning. By quantitative reasoning, we mean reasoning on basis of the formula $d\, sin\, \theta = m\lambda$. This formula shows how the path length difference is related to the slit separation and the angle to a location on the screen (i.e., $\Delta D = d\, sin\, \theta$). The path length difference is written in terms of wavelength (on the right-hand-side of equation $d\, sin\, \theta = m\lambda$). Therefore, the value *m* provides information about the location on the screen. A student, for example, answered that, *"a minimum, $sin\, \theta$ is constant so $d/2\, sin\, \theta = m\lambda/2$. For B, $m = 1$ so $d\, sin\, \theta = \lambda/2$ corresponds to a minimum at B."* In this example, the student implicitly started with formula $d\, sin\, \theta = m\lambda$. The student correctly recognized that $sin\, \theta$ is held constant, and the right-hand-side of the equation would change when the slit separation is changed. The student then identified that $m = 1$ for point B (with the original slits). Finally, the student determined that with four slits the right-hand-side becomes $\lambda/2$, which corresponds to a minimum for four slits.

6.4.3.a.3 Memorized rules

Some students made use of memorized rules as part of their explanation. As mentioned in the first approach, some students used a rule for the possible values of path length difference (between adjacent slits) for minima with four equally-spaced slits. The percentage varies among tasks since the slits are not equally-spaced on both versions of Task 6.4. Therefore, we do not discuss the prevalence of this rule.

It is worthwhile to mention that on all tasks, approximately 10% of the students used a memorized rule that applies only in certain cases used in the tutorial. However, this rule does not apply in the cases on Task 6.3 and Task 6.4, therefore these students arrived at incorrect answers. We will discuss in detail how student arrived at their answers based on the memorized rule in 6.4.3.b.

About another 15% of the students did not use any of the approaches mentioned above. They either did not give explanations or had explanations that were not well-articulated. We consider these students and those who misused the memorized rule not applying the principle of superposition.

In summary, among all the responses, we found that about 75% of the students applied superposition to some extent. Very few students explicitly discussed the values of phase difference. Instead, students typically discussed qualitatively whether the waves are in phase or out of phase. Many students explained on the basis of qualitative reasoning about path length difference. Some gave quantitative reasoning based on the formula.

6.4.3.b Identification of student difficulties

For each of the approaches mentioned above, we identify specific difficulties students had. Overall, we found these difficulties representative and can provide insights into curriculum development. However, we do not intend to report the prevalence for each difficulty since the extent to which students gave thorough explanations varies.

The first three difficulties all correspond to the first approach. The fourth and fifth difficulties correspond to the second and the third approaches, respectively.

6.4.3.b.1 Tendency to add effects of superposition for pairs of waves

We found that many students tended to reason about the superposition of the four slits based on adding the effects of superposition for pairs of waves. This type of reasoning can be productive in some cases, but is not generalizable. We illustrate this type of reasoning using an example response: *"B is a maximum b/c [because] [slits] 1+2 are a max and [slits] Y+Z have the same separation d, and are a max at θ, so adding them [max] up is a maximum."* This student correctly explained that light from slits 1 and 2 maximum constructively interfere and that same holds true for light from slits Y and Z because they have the same slit separation. Instead of discussing whether waves through slits 1 and 2 are in phase or out of phase with waves through slits Y and Z, the student then added incorrectly the effects of superposition: *"adding them [max] up is a maximum."*

Other students who used path length difference as part of their explanation often only focused on the effect of superposition for the pairs of slits. For example, *"Point B is a principal maximum because $\Delta D_{12} = \lambda$ and $\Delta D_{ZY} = \lambda$, so the wave will all constructively interfere."* This student also only focused on the fact that the waves in pairs are in phase and incorrectly concluded that all waves constructively interfere.

6.4.3.b.2 Tendency to add effects of superposition for all pairs of waves

The students above who added effects of superposition for one specific pairing of slits, some considered all possible pairing. They typically arrived at a different answer. For instance, *"Neither, while 1 & 2 do have a $\Delta\mathrm{d}$ of $1\ \lambda$ there, 1 & y have a $\Delta\mathrm{d}_{adj}$ of $0.5\ \lambda$ (similar triangles), so do the pairs y & 2 and 2 & z so it can not be a principal maximum."* This student correctly determined the path length difference for each pair of slits. This student then reasoned that some pairs maximum constructively interfere and some destructively interfere, and decided it is neither a principal maximum or minimum. This student did not seem to recognize that since waves through slits 1 and Y completely destructively interfere and waves through slits 2 and Z completely destructively interfere as well, it does not matter whether waves slits 1 and 2 are in phase or out of phase.

6.4.3.b.3 Tendency to add path length difference for pairs of slits

We also found that some students added the path length differences together and used incorrectly the "sum" to determine the interference pattern. *"Now, ΔD between adjacent slits is $\lambda/2 \Longrightarrow \frac{\lambda}{2}+\frac{\lambda}{2}=1\lambda$ for [slits] 1+y and [slits] 2+z. This means there is constructive interference at [point] B, a max [maximum]."* Although this type of error was not very common, we thought it was worth mentioning because this is another example in which students were able to determine the path length differences but used them incorrectly. The first three difficulties illustrate different ways students used path length differences. These results suggest that many students were able to determine the path length differences, but they unsuccessfully used them to infer the superposition of all waves.

6.4.3.b.4 Tendency to manipulate formula without conceptual reasoning

We identified some responses in which students incorrectly manipulated the formula $d\, sin\, \theta = m\lambda$. As discussed in 6.4.3.a, this formula shows how the path length difference (written in terms of wavelength on the right-hand-side of the equation) is related to the slit separation *d* and the angle θ to a location on the screen. To use the formula correctly, students were required to use conceptual reasoning to some degree. In particular, students need to recognize the independent and dependent variables in the corresponding physical scenarios. Those who incorrectly manipulated the formula did not seem to incorporate conceptual reasoning. For example, one student answered, *"$d \sin\theta = \lambda$ becomes $d/2 \sin\theta = \lambda$ or $d \sin\theta = 2\lambda$. This is, principle max as well, with n corresponding to $n = 2$ instead of $n = 1$."* This student correctly recognized that the slit separation is halved. However, the student incorrectly held the path length difference (right-hand-side of the equation) constant. The student then multiplied the equation by 2 on both sides and arrived at $d \sin\theta = 2\lambda$. Lastly, the student recognized that (a path length difference of) 2λ corresponds to a principle maximum. In contrast, the student who correctly used this formula (see 6.4.3.a) were able to recognize that the path length difference changes when the slit separation changes with θ holding constant.

This type of error was not very common due to (at least in part) that the questions were qualitative rather than quantitative. However, it is worth discussing since it provides insights into how students make sense of formula and how they relate formalism to physical phenomena. The results suggest that manipulating formula successfully requires conceptual reasoning. Students who correctly identified the dependent and independent variables were more likely to manipulate formula successfully.

6.4.3.b.5 Tendency to use memorized rule

Some students use a rule that they discover from the cases presented in the tutorial but that does not apply to the tasks given. In the tutorial, students consider cases in which slits are added such that the distance between the adjacent slits does not change and they discover that the principal maximum would always remain a principal maximum. However, this rule does not apply to Task 6.3 and Task 6.4 since the slit separation changes. *"Since additional slits do not change the locations of principal maximums (always at $n\lambda$, where $n = 0, 1, 2, \ldots$) and B is a mximum with*

2 slits, B will remain a maximum with 4 slits." This student used a memorized rule without justifying the conditions that the rule applies and arrived at an incorrect conclusion.

6.4.4 Summary

In section 6.4, we discussed an investigation into student ability to apply superposition to multiple-slit interference. The tasks we designed involved scenarios in which students considered a few locations on an interference pattern resulted from two slits and they were asked to decide how the pattern would change when two additional slits were added. Overall, student did not perform well.

We examined in detail the approaches students used to reason about multiple-slit interference. About 75% of the students used superposition to some extent. They discussed the path length difference, qualitatively described phase difference (i.e., in phase or out of phase), and/or made used of formula. The rest of the students (about 25% of all students) either relied on memorized rule that does not apply or did not provide much detail about their reasoning. The results suggest that many students do not use superposition principal when they reason about multiple-slit interference.

We have also examined the extent to which students are able to use path length difference and/or phase difference to reason superposition. We found that students tend not to use phase difference explicitly but rather to qualitatively describe whether the waves are in phase or out of phase. The results also suggest that many students although are able to determine correctly the path length difference, do not use it successfully. In particular, many students tend to add *effects* of superposition of pairs of waves, or of all pairs of waves; some tend to add the path length difference for pairs of slits.

Our results also show that some students use formula as part of their reasoning. Students who use it correctly tend to involve conceptual reasoning when manipulate the formula. Students who use it incorrectly do not seem to recognize the independent and dependent variables in the physical scenarios.

6.5 Summary

This chapter discussed an investigation into student ability to reason with classical interference. In particular, the investigation focused on student functional understanding of phase difference and path length difference. The contexts involved in the investigation were two-source interference of water waves and multiple-slit interference of light.

To probe student functional understanding of phase difference and path length difference, we designed tasks in which two sources of water waves are 180° out of phase. The tasks prompted students to determine the phase difference or path length difference along a particular line, and the source separation. The results suggest that many students do not differentiate between phase difference and path length difference. Some express phase difference in terms of wavelength. In addition, many students do not treat path length difference and source separation as purely length quantities. Many "add" the phase difference between the sources to the path length difference (and to the source separation). The findings suggest that many students conflate the concepts of phase difference and path length difference.

We also investigated in detail how students apply superposition to multiple-slit interference. In particular, we examined the approaches student used and the extent to which they were able to use them. We found that about three-quarters of the students use superposition. These students tend to discuss the path length difference, qualitatively described phase difference (i.e., in phase or out of phase), and/or make use of formula. About one-quarter of the students either rely on memorized rule that does not apply or provide little detail about their reasoning. The results suggest that after lecture and tutorial instruction, there is still a fraction of students do not apply the principal of superposition when they reason about multiple-slit interference.

When students reason about superposition, many are able to determine the path length difference and to superimpose waves in pairs, but unsuccessfully determine the interference result for all waves. For example, many students tend to reason that if a pair of waves are in phase and another pair of waves are in phase, then all the waves must be in phase. This type of reasoning is not always correct since one pair of waves can be completely out of phase with another pair of waves, resulting into a point of zero intensity. Other students tend to reason that some pairs of waves maximum constructively interfere (e.g., 1 and 2 maximum constructively interfere, 3 and 4 maximum constructively interfere), but some completely destructively interfere (e.g., 1 and 3 completely destructively interfere, 2 and 4 completely destructively interfere), so neither a principal maximum or minimum. These students do not seem to recognize that two waves completely cancel with another two waves. Although not very common, some students tend to add the path length difference and then use the *total* path length difference to determine the interference pattern. It seems that these students attempt to determine the net effect of superposition but use an incorrect strategy.

The results discussed in this chapter suggest that the concept of phase difference is abstract and challenging for many students. Many have difficulty with using phase difference or path length difference to reason about superposition even after tutorial instruction. We have been developing curriculum to address these difficulties. In particular, the tutorial we developed makes use of a new representation, phasors, which we thought could facilitate students to visualize superposition. In chapter 7, we discuss in detail how the curriculum has been developed. Results from a preliminary assessment of the curriculum are also discussed.

Chapter 7. Development and Assessment of Tutorial on Phasors

The results in chapter 6 suggest that the concept of phase difference is challenging for many introductory-level students. Many students confuse phase difference with path length difference. These students do not always treat path length difference as a length quantity or recognize that phase difference does not have units. For instance, when two sources are 180^{o} out of phase, many students tend to add half of the wavelength to the path length difference and to the source separation without recognizing that those quantities are independent of the phase difference between the sources (see section 6.3). We have also found that students have difficulty in reasoning about superposition of more than two waves. For example, when reasoning about multiple-slit interference, many students are able to superimpose waves in pairs using path length difference (or phase difference), but unsuccessfully superimpose all the waves (see section 6.4).

To help students better understand phase difference and address the difficulties students have with interference, we developed a tutorial on phasors, which was added to the existing sequence of tutorials on interference and diffraction (see section 7.1). Although the original sequence of tutorials does not use phasors, this representation is commonly introduced by many textbooks and instructors. (The textbooks used during this research project all had sections covering phasors in physical optics.) When representing waves as phasors, each wave is drawn as an arrow. The phase difference between two waves at a given point on a screen is indicated by the angle between the arrows. The phasor representation has some affordances: (1) phasors support the visualization of superposition through the relatively easy addition of directed arrows and (2) phasors can readily be used to find the intensity of light at any point on a screen, not only at the principal maxima and minima.

Considering these affordances, we thought that making use of phasors might be a productive instructional strategy. Especially since prior research has shown that under certain circumstances, the phasor representation can be effective at supporting high school students' conceptual understanding of interference [74]. On a survey involving conceptual questions about interference, students who had been taught phasors outperformed students who had been taught to represent light waves as sinusoidal curves.

In this chapter, we start by describing tutorial *Phasors* in section 7.1. Section 7.2 presents results from an assessment of the use of the new sequence of tutorials on interference. This chapter ends with a summary in section 7.3.

7.1 Development of Tutorial *Phasors*

As discussed in chapter 1 (section 1.2.2), the tutorial section of the introductory waves and optics course (PHYS 123) often starts with two tutorials on superposition of waves and the boundary behaviors. A sequence of four tutorials on interference and diffraction follows.

The latter sequence starts with *Two-source interference*, in which students are guided to reason about superposition and interference of water waves. *Wave properties of Light* is intended to help students to recognize the wave behavior of light through drawing analogy from water waves. The third tutorial in the sequence is *Multiple-slit interference.* This tutorial provides opportunities for students to consider how the intensity of particular points on a screen with two-slit interference would change when one or more slits are added. Students are then provided a set of transparencies of sinusoidal curves to identify the path length difference between adjacent slits for three or four slits. At the end of this tutorial, students are guided to reflect on and generalize the ideas they have developed in this tutorial. The last of the sequence is *A model for single slit diffraction,* which is a continuation of the first three tutorials. Students are expected to model a single slit as infinite number of small slits and to use the ideas developed from the previous tutorials to reason about diffraction. Path length difference and the resultant phase difference are key elements in the models of the entire sequence of tutorials.

In tutorial *Multiple-slit interference,* students determine the values of phase differences based on the values of the corresponding path length differences, which we thought is a more natural approach as opposed to introducing the concept of phase difference before path length difference. We decided to design the tutorial *Phasors* such that it builds on the ideas developed in *Multiple-slit interference*. In particular, the tutorial *Phasors* guides students to use the values of phase differences (determined based on the corresponding path length differences) to sketch phasor diagrams for multiple-slit interference. Therefore, we placed the tutorial *Phasors* after tutorial *Multiple-slit interference*. As this research project proceeded and the learning goals being refined, an exercise included in (one of the versions of) the tutorial homework *Phasors* was moved to tutorial homework *Two-source interference* in order to better fit into the context.

In this section, we describe the new tutorial *Phasors*. In section 7.1.1, we start with the initial development of the tutorial. The modified version of the tutorial is discussed in section 7.1.2. Lastly, we describe the modified tutorial homework *Two-source interference* in section 7.1.3.

7.1.1 Initial development of Tutorial and homework *Phasors*

The initial version of tutorial *Phasors* is described in 7.1.1.a, followed by the initial version of tutorial homework *Phasors* in 7.1.1.b. The tutorial is intended to follow instruction on phasors in lecture and/or the textbooks.

7.1.1.a Initial version of Tutorial *Phasors*

The first section of the tutorial (see Appendix A, Figure A.18) is intended to help students recognize that phasors can be treated as rotating vectors and that the phase difference between two sinusoidal functions can be represented by the angle between the associated phasors. At the beginning of the first section, students consider a unit vector $\vec{v}$ rotating in the *x*-*y* plane. Students are asked to write an expression for the *y*-component of the unit vector v_y as a function of time and to describe how it changes as time increases. They are expected to recognize that v_y behaves like a sinusoidal function. Students are then guided to sketch another vector $\vec{u}$ such that the vector sum of $\vec{u}$ and $\vec{v}$ is zero at all time instants. They are expected to recognize that the angle between the two vectors is π and does not change in time. At the end of the first section, students are prompted to use their expression for v_y and the angle to write expression for the *y*-component of the unit vector u_y.

The second section of the tutorial provides students an opportunity to apply phasors to the context of water waves with two sources S_1 and S_2. The primary objective is to help students recognize that the phasors (and the waves) change in both time and space. In part A, students first consider point A, which lies on a line of maximum constructive interference. The displacement versus time graph at point A due to source S_1 only is given. Students are asked to sketch the displacement over time at point A due to source S_2 only. Students are then asked to sketch a pair of phasors to represent the displacement at point A due to each of the two sources at a particular time. Lastly, students are asked to determine the angle between the phasors and to consider whether this angle would be different if another point along the line is considered. They are expected to recognize that the phasors would always point in the same direction for any point along the line of maximum constructive interference. In part B, students are guided to repeat this exercise for a point (point B) that lies on a nodal line. They are expected to recognize that the phasors at point B always point in the opposite directions. In part C, students consider an entire nodal line at a single instant in time. Again, they are given the displacement graph along this nodal line due to source S_1 only. On the graph, point B marks a point that has the largest displacement below the equilibrium, while point C marks a point is at the equilibrium. They are asked to sketch phasor diagrams for two these two points. Based on our observation in class, this actually serves a usful purpose. Many students incorrectly assume that the phasors have zero magnitudes at point C since it is at the equilibrium. Moreover, many students have difficulty labeling the phasor due to each source at point C. Those students usually fail to consider how the displacement due to each source changes in time at point C. The tutorial is intended to help students go through the necessary reasoning.

In the last section of the tutorial, students consider phasors for an interference pattern on the screen due to multiple slits. In part A, students are asked to sketch phasors for a principal maximum with three equally spaced narrow slits. They are then asked to determine the angle between adjacent phasors (phasors that correspond to adjacent slits). They are expected to recognize that any multiple of 2π is possible. In part B, students are asked to sketch phasors that correspond to a minimum with three slits on the screen and to determine the angle between adjacent phasors. Students are then asked to write out the first few values of the phase difference between adjacent phasors. At the end of the section, students are prompted to sketch phasors to represent minima for cases involving four, five, and six slits. They are given an opportunity to generalize the pattern they observed.

7.1.1.b Initial version of tutorial homework *Phasors*

The homework (see Appendix B, Figure B.6) involves two problems. In the first problem, students consider two waves with the same amplitude and wavelength traveling in the opposite directions on a spring. The displacement as a function of time for one wave is given, students are asked to write the displacement as a function of time for another wave. They are then given the expressions for two different positon on the spring and are asked to sketch the phasor diagrams at several different time instants for each of these positions. Students are prompted to describe how the new displacement of the spring at those positions change in time using their phasor diagrams. We expect students to recognize that one of them is a node that is always at the equilibrium and the other is an antinode at which the displacement changes the most. Toward the end of this problem, students are asked to sketch the phasor diagrams at several positions but at the same instant in time. Finally, students are given an opportunity to reflect on what they did in this exercise. They are expected to use their phasor diagrams to justify why the net displacement formed by the two waves is called a standing wave.

The second problem is intended to help students to use phasor diagrams to determine the intensity of a secondary maximum with three slits of equal distances apart. Students first consider the minima between two adjacent principal maxima. They are asked to write out values of the phase difference between adjacent slits that correspond to these minima. They are then told that the phase difference between adjacent slits for the secondary maximum is half-way between the values they wrote previous. They are asked to sketch phasors that corresponds to the secondary maximum. Lastly, they are guided to use vector sum to determine the intensity for the secondary maximum.

7.1.2 Modifications to Tutorial *Phasors*

Based on our classroom observation, the ideas developed in the initial version of tutorial *Phasors* seemed straightforward to many students. Thus, the difficulty level of the tutorial needed to be increased. We decided to design more exercises about points on a screen that are neither principal maxima nor minima. This facilitates students to reason about interference in more general cases rather than focusing on a few special cases, such as principal maxima and minima.

7.1.2.a Modified tutorial *Phasors*

The modified version of the tutorial (see Appendix A, Figure A.21) is comprised of four sections. In the first section, a scenario involving a spring-block system is added after students consider a rotating vector. Students are guided to relate the *y*-component of the vector to the simple harmonic motion of the block. They are expected to recognize that the *y*-component of the vector as a function as time can be used to describe the simple harmonic motion and the *x*-component can be ignored. At the end of the first section, the concepts of phasor and phase are introduced.

In the second section of the tutorial, students consider addition of phasors. Instead of having two vectors pointing in the opposite directions as in the initial version, the angle between vectors $\vec{u}$ and $\vec{v}$ is smaller than 90 degrees. These two vectors rotate in the same direction with the same angular frequency. Students are asked to sketch the vector sum $\vec{s}$ at a particular instant in time. They are then asked to compare the angular frequency of $\vec{s}$ to the angular frequency of $\vec{u}$ and $\vec{v}$. They are expected to recognize that the angle between $\vec{s}$ and $\vec{u}$ (or $\vec{v}$) is constant in time, and thus all three vectors have the same angular frequency.

The third section, which is in the context of water waves, now only contains the parts in which students think about how phasors change in time at particular locations. The part that involves how phasors change in space at the same instant in time has been removed. This decision has been made based on two facts: many students spend a long time on this part and the idea is not one of the most important.

Several changes have been made to the last section of the tutorial. An interference pattern on the screen with three slits is provided to students. This allows students to relate the phasor representation to the diagram for specific points on screen. We have also added more guiding questions to help students recognize that the relative phase (between adjacent slits) increases monotonically when the point moves from the center of the screen to each side. We believe this can help students recognize the correspondence between a specific value of relative phase and a specific point on screen. Questions that prompt students to consider the difference between the phasor diagrams (or phase differences) corresponding to two adjacent minima are added since we have found that many students incorrectly assume they are the same. An exercise about the secondary maximum in the homework of the initial version has been moved to the modified tutorial. Moreover, a question that asked for the ratio between the intensity of the secondary maximum and the intensity of the principal maximum has been added.

7.1.2.b Modified homework *Phasors*

Substantial changes have been made to the homework (see Appendix B, Figure B.9). The question on standing waves has been removed since the preliminary results of our assessment suggest that students need more exercises on interference of light. Therefore, we designed four problems that support students to use phasors to reason about interference.

In problem one, students consider three phasor diagrams. The relative phase between adjacent slits is the same for all three cases, but each case has a different number of slits. Students are asked to determine the number of slits for each case. They are then asked to rank the cases according to the intensity. Lastly, students are told that in each case one of the slits is covered and they are asked how the intensity for each case would change.

In problem two, students consider a three-slit (equally-spaced) interference pattern on a screen. The phasor diagrams for two different points on the screen and the corresponding values of the relative phase are given. Students are asked to mark the locations on the interference pattern. They are then prompted to consider how the intensity for each location would change if a fourth slit is added while the distance between adjacent slides is the same as before.

Problem three is intended to provide students an opportunity to sketch the phasor diagrams for minima with four equally-spaced slits. This problem has been added because many students overgeneralize the pattern that they have observed in the tutorial that both phasor diagrams for the minima of three slits can form triangles. This problem prompts students to check whether their phasor diagrams are consistent with the values of phase difference between adjacent slits. They are expected to recognize the inconsistency if they incorrectly assume that the phasor diagram for each of the three minima can form a square.

In the last problem, students consider an interference pattern on screen with two slits S_1 and S_2. They are told that another slit S_3 is added such that the distance between S_3 and S_2 is half of the distance between S_1 and S_2. They are then guided to use the phasor diagram to determine how the intensity would change for two different points on screen.

7.1.3 Modifications to Tutorial Homework *Two-source Interference*

To help students differentiate between the path length difference and the phase difference, we designed a homework problem in the context of water waves with two point sources. This problem, originally included in tutorial homework *Phasors,* has been added as the last problem in the homework (see Appendix B, Figure B.10) of tutorial *Two-source interference*.

In this problem, students consider two cases. The interference patterns with nodal lines and lines of maximum constructive interference are given. In each case, students first consider whether the sources are in phase or out of phase; they are then prompted to determine the path length difference and the phase difference for a nodal line and a line of maximum constructive interference, respectively. They are asked to compare the values in the two cases. They are expected to recognize that the path length difference along the same line of locations (whether or not it is a nodal line or a line of maximum constructive interference) would be same for both cases, while the phase differences are different. The last question is intended to help students reflect on and generalize the ideas they have developed in this problem. They are told to consider another two sources and that the path length difference for a particular point is one wavelength. They are asked if it is possible to tell whether this point lies on a nodal line or a line of constructive interference.

7.2 Assessment of the New Sequence of Five Tutorials on Interference

This section presents results from a preliminary assessment of the new sequence of tutorials on interference. We compare the performance of classes of students who had the initial sequence of tutorials and the performance of classes of students who had the new sequence of tutorials (with the modified version of tutorial *Phasors* rather than the initial version). Note that none of the students included in this chapter had worked through the problem described in the previous section that was added to the tutorial homework *Two-source interference*.

All the tasks used for assessment were given on exams after students received relevant lecture instruction and completed the entire sequence of tutorials (either the initial or the new sequence). Some were given on midterm exams, and some were given on final exams. Based on the course policy, the tutorial-related questions on midterm exams were in long-response format (explanation required) by default, and all the questions on final exams were in multiple-choice format. However, in one academic quarter, we administered tasks on final exams in long-response format (see section 7.2.2).

This section has three subsections. In 7.2.1, we present results from an assessment of student functional understanding of path length difference. In 7.2.2, we discuss results from an assessment of student ability to apply superposition to multiple-slit interference. Lastly, 7.2.3 shows results from an assessment of student ability to relate intensity to phasor diagrams.

7.2.1 Assessing student functional understanding of path length difference

Task 6.2 (see Figure 6.2 in chapter 6) was used to assess the new sequence of tutorials on student functional understanding of path length difference. Recall that in Task 6.2, students consider water waves due to two point sources. A top view diagram with nodal lines and lines of maximum constructive interference are given. Students are told that the sources are 180^{o} out of phase. They are asked to choose the answer that best describes (1) the path length difference for each line and (2) the source separation, respectively.

Task 6.2 was given on final exams to classes of students who either completed the initial sequence of tutorial or the new sequence of tutorial (with the modified version of *Phasors*). Student performance on each of the question of Task 6.2 is shown in Table 7.41.

Table 7.41. Comparison of student performance on Task 6.2

Questions	Path length difference	Source separation
Without tutorial *Phasors*	58% (N = 395[a])	54% (N = 395[a])
With tutorial *Phasors*	57% (N = 375[b])	48% (N = 179[c])

[a]PHYS 123B: WIN 13. PHYS 123C: SPR 13. [b]PHYS 123B: WIN 18. PHYS 123D: SPR 18. [c]PHYS 123D: SPR 18

The results from Fisher's exact tests suggest that the difference on both questions are not statistically significant ($p > 0.05$). This may not be surprising since the *Phasors* tutorial does not explicitly address the difficulty with conflating path length difference and phase difference. (This was the motivation for modifying the tutorial homework *Two-source interference* described in section 7.1.3, in which students consider a case with two sources in phase and a case with two sources that are completely out of phase. We are planning to assess the impact of the modified tutorial homework in the future.)

7.2.2 Assessing student ability to apply superposition to multiple-slit interference

Here, we discuss the assessments of student ability to apply superposition to multiple-slit interference. The assessments were conducted on final exams[4] after students completed the new sequence of tutorials in two academic quarters. In Autumn 2017, we used question 2 (point B) of Task 6.4 (see Figure 6.3) to examine in detail how the new sequence of tutorials can affect student performance and reasoning. We chose question 2 because point B seemed the most challenging for students based on results from chapter 6. The task was administered on final exams in long-response format. In Spring 2018, we gave the entire Task 6.4 on final exams in multiple-choice format. Therefore, only student answers (not reasoning) were evaluated. In both quarters, we used data (discussed in chapter 6) collected from classes in which students had the initial sequence of tutorials for comparison. Recall that in those classes, the task was given on midterm exams in long-response format.

7.2.2.a The first assessment: examining the impact on student reasoning

Figure 7.1 shows student performance on different versions of Task 6.4, question 2. The darker shade shows the percentage of correct answer with correct reasoning, and the lighter shade shows the percentage of correct answer without correct reasoning. The results from fisher's exact tests suggest that on both versions, students who had the new sequence of tutorials performed significantly better than students who had the initial sequence of tutorials. For both correct answer and correct answer with correct reasoning, the difference is statistically significant ($p < 0.001$).

[4] In those academic quarters, there were two midterm exams. The second midterm occurred before students completed the entire sequence of tutorials. To assess the impact of the entire sequence, we administered tasks on final exams. However, all the questions on final exams were by default multiple-choice format.

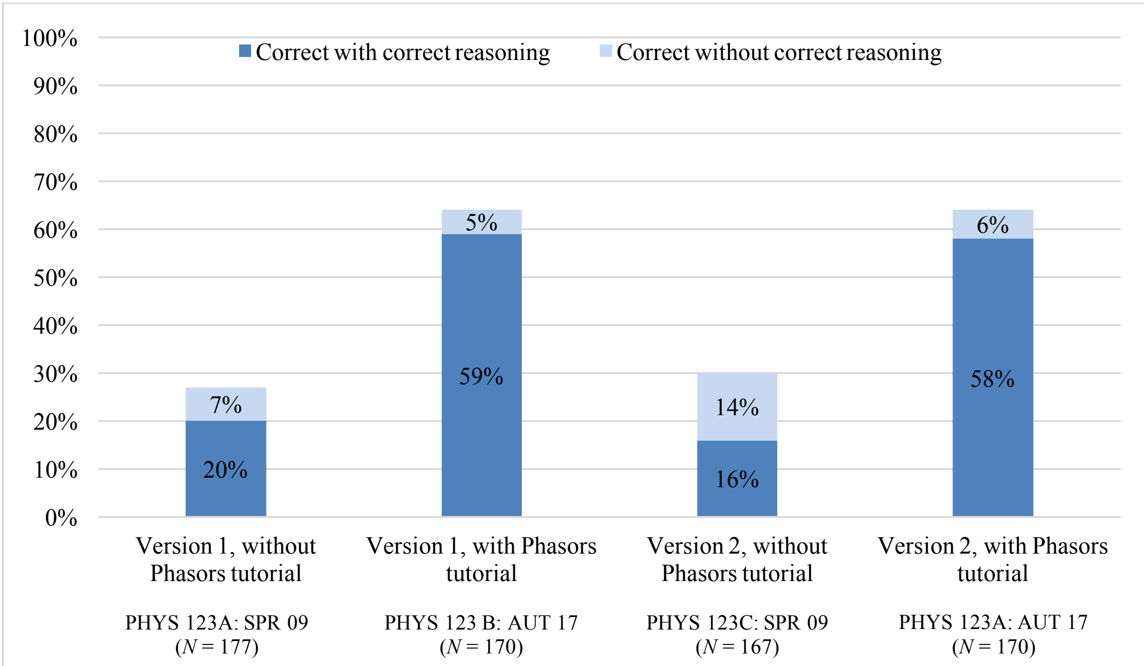
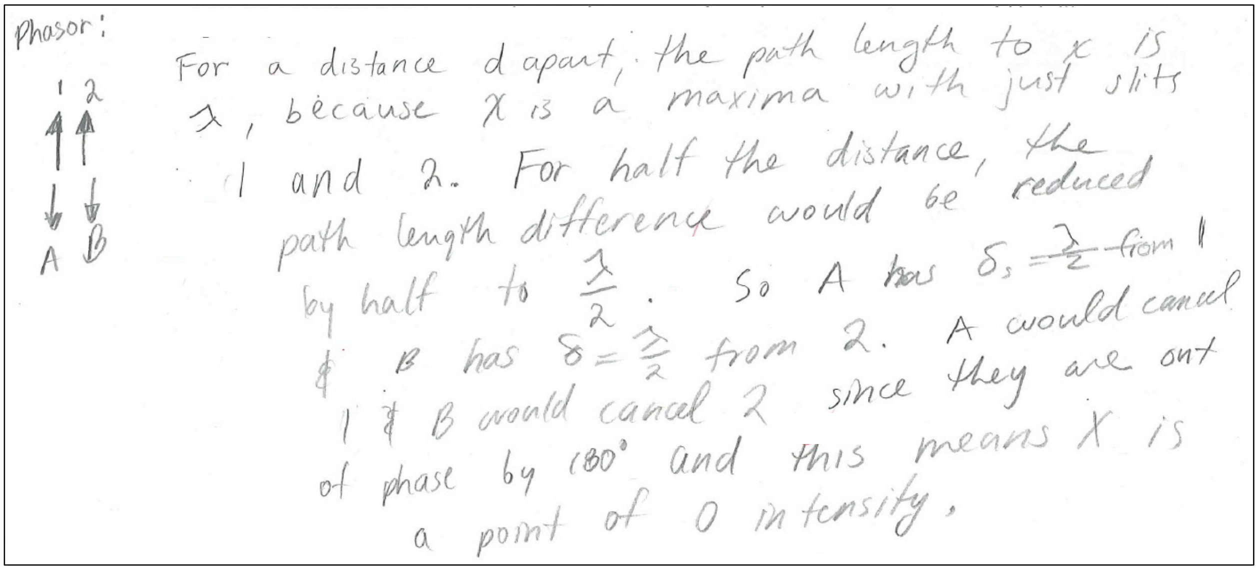


Figure 7.1. Comparison of student performance on Task 6.4, question 2.

For students who had the new sequence of tutorials, about 85% qualitatively or quantitatively used superposition. Recall that only about 70% of the students who had the initial sequence of tutorials used superposition, as discussed in chapter 6. It seems that the new sequence of tutorials supports more students to apply superposition.

Even without giving students a specific prompt to use phasors, about 20% of the students who had the new sequence of tutorials used phasors on this question. Most of these students used phasors correctly. An example response (to version 1) is shown in Figure 7.2. This student first correctly determined the path length difference, and then correctly determined the corresponding phase difference. Finally, the student successfully superimposed all the waves using a phasor diagram.


Phasor:

1 2

A B

For a distance d apart, the path length to x is λ, because x is a maxima with just slits 1 and 2. For half the distance, the path length difference would be reduced by half to $\frac{\lambda}{2}$. So A has $\delta_1 = \frac{\lambda}{2}$ from 1 & B has $\delta = \frac{\lambda}{2}$ from 2. A would cancel 1 & B would cancel 2 since they are out of phase by 180° and this means X is a point of 0 intensity.

Figure 7.2. An example student response to question 2 of Task 6.4, version 2.

The results suggest that the new sequence of tutorials appears to be effective at supporting students to apply superposition to multiple-slit interference. We have found that more students used superposition principal after they completed the new sequence of tutorials. Students who completed the new sequence of tutorials outperformed significantly students who had the initial sequence of tutorials. Moreover, many students spontaneously used phasor diagrams as part of their explanations and most of them used successfully.

7.2.2.b The second assessment: examine the impact on overall performance

The results from the first assessment suggest that the new sequence of tutorials have a positive impact on student reasoning. In this section, we report data from the second assessment, in which we compare student performance on the entire set of questions on Task 6.4. Recall that in classes in which students who had the new sequence of tutorials, Task 6.4 was administered on final exams in multiple-choice format. In classes in which students had the initial sequence, Task 6.4 was given on midterm exams in long-response format.

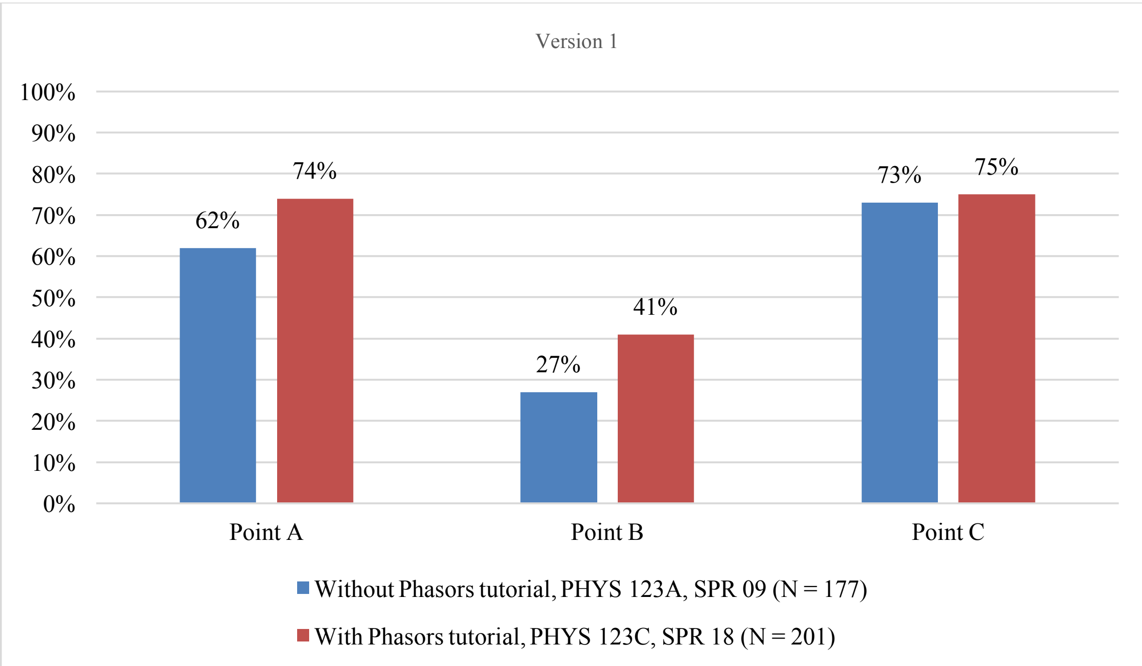


Figure 7.3. Comparison of student performance on Task 6.4, version 1.

Figure 7.3 shows the percentage of correct answer for all three questions of Task 6.4, version 1. For points A and B, students who had the new sequence of tutorials did better than students who had the initial sequence of tutorials. The differences for both points are statistically significant ($p < 0.05$ for point A and $p < 0.01$ for point B). For point C, the difference is not statistically significant. This result is likely due to that many students already did very well after they completed the initial sequence.

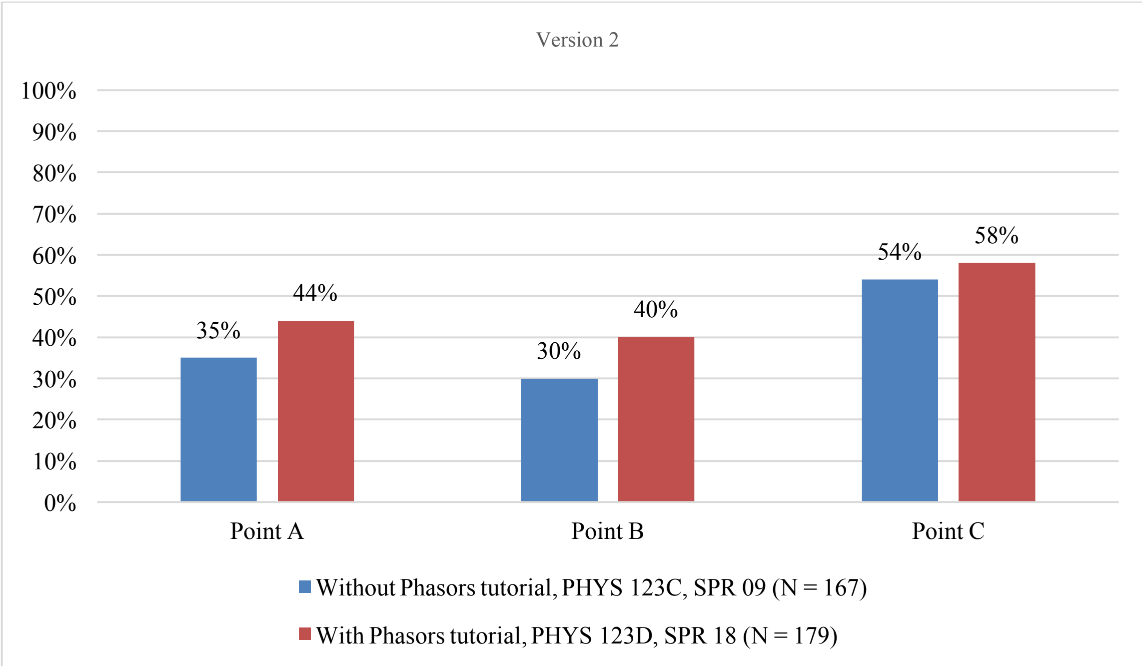


Figure 7.4. Comparison of student performance on Task 6.4, version 2.

Figure 7.4 shows the percentage of correct answer for all three questions of Task 6.4, version 2. For all three points, the differences in percentages of correct answer between students who had the new sequence and students who had the initial sequence are not statistically significant ($p = 0.079$ for point A, $p = 0.071$ for point B, and $p = 0.450$ for point C).

7.2.2.c Summary

The data collected from the first assessment (in Autumn 2017) show that students who completed the new sequence of tutorials significantly outperformed students who had the initial sequence of tutorials. A significantly greater portion of students gave correct answers, and a significantly greater portion of students gave correct reasoning. Moreover, the new sequence of tutorials seemed to have a positive impact on student approaches as well. A greater percentage of students who had the new sequence of tutorials used superposition principal compared to students who had the initial sequence. Many of the students who had the new sequence spontaneously used phasor diagrams correctly as part of their explanations. However, the data from the second assessment (in Spring 2018) did not show comparable improvement. On one version, the improvement was statistically significant, while on another version, the improvement was not. To try to understand this difference, we compared the conditions under which those data were collected in each of the quarters. Several differences were identified that we speculate might have influenced the results.

The lecture instructor in Autumn 2017 reviewed tutorial homework in class on a regular basis. Although the instructor did not teach phasors in lecture, he encouraged students to use phasors. We speculated that this might have a positive impact on the approaches that students used. The fact that many students (about 20%) spontaneously used phasors appears to be consistent with our speculation.

In Spring 2018, the entire set of Task 6.4, rather than one question, was given. It seems likely that the other questions influenced student reasoning. For example, students who only considered the interference of waves through the original slits and the interference of waves through the additional slits would arrive at a correct answer on question 1 (in which point A is a minimum with the original slits) but would likely arrive at an incorrect answer in question 2 (in which point B was a maximum the original slits). This might have resulted in a smaller amount of improvement of performance in Spring 2018.

Task 6.4 was given as multiple-choice format with no explanation required in Spring 2018, while explanations were required in the other classes, including those in which students had the initial sequence of tutorials. The multiple-choice format was likely to have triggered more intuitive but incorrect responses. Ives and Stang show that asking for an explanation on the multiple-choice exam questions can engage reflective thinking and can improve student performance [78].

The hypotheses above are only possible ways to accounting for the difference in results we observed. We are planning to collect more data for assessment and will modify the tutorials if necessary.

7.2.3 Assessing student ability to relate intensity to phasor diagrams

Figure 7.5 shows two tasks on relating intensity to phasor diagram. The tasks were designed by a lecture instructor who taught phasors in depth. We used these tasks to assess the effectiveness of the new sequence of tutorials at facilitating student ability to relate intensity to phasor diagrams. We compared classes in which students had the same lecture instructor but received different tutorial instruction, either the initial or the new sequence. The tasks were administered on final exams in all the classes.

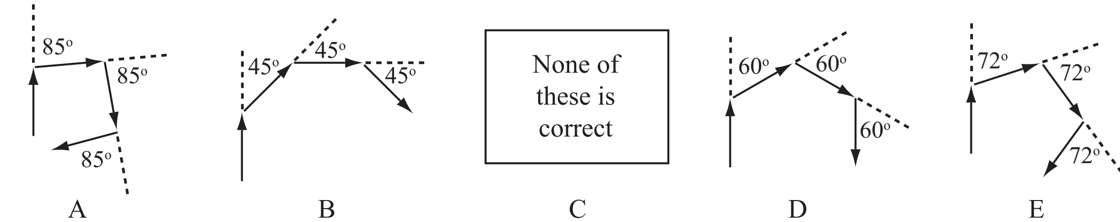
In Task 7.1, students consider interference with four equally-spaced very narrow slits. They are asked to choose a phasor diagram that best corresponds to a location on a screen where the intensity is one sixteenth of the intensity at the center of the screen. Similarly, in Task 7.2, students consider interference with five equally-spaced very narrow slits. They are asked to choose a phasor diagram that best corresponds to a location on a screen where the intensity is one twenty-fifth of the intensity at the center of the screen.

To answer, students need to recognize that the vector sum of the phasors corresponds to the amplitude of the resultant wave and intensity is proportional to the amplitude squared. Therefore, the vector sum of four phasors at the specified location in Task 7.1 should be one-fourth of the vector sum of phasors that corresponds to the center of the screen. At the center of the screen, it is a principle maximum. For a principle maximum, all the phasors point in the same direction. Thus, the vector sum should have a magnitude that is equal to the magnitude of a single phasor. Answer E is the correct answer in Task 7.1. Following the same logic, the correct answer in Task

7.2 is answer B. In both tasks, answer A would likely be chosen by students who do not recognize that intensity is proportional to the amplitude squared. The other phasor diagrams correspond to points that have intensity greater than the value given in the prompt of each task.

Task 7.1

A set of four equally-spaced very narrow slits is used to produce a pattern of light on a screen.
Which choice best represents a phasor pattern appropriate to represent a location where the intensity on the screen is one sixteenth (1/16) the intensity at the center of the screen?



Task 7.2

A set of five equally-spaced very narrow slits is used to produce a pattern of light on a screen.
Which choice best represents a phasor pattern appropriate to represent a location where the intensity on the screen is one twenty-fifth (1/25) the intensity at the center of the screen?

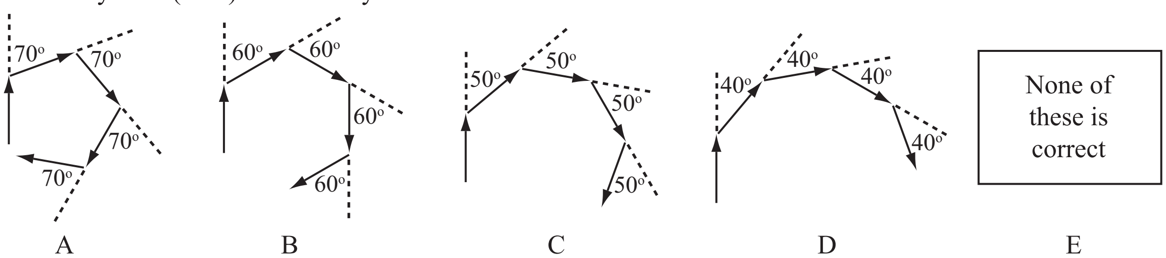


Figure 7.5. Tasks on relating intensity to phasor diagram

Student performance on Task 7.1 is shown in Table 7.42. For students who had the initial sequence of tutorials, only about 26% of the students chose the correct answer. For students who had the new sequence of tutorials, about 39% of the students answered correctly. The task seems challenging for students. The difference in prevalence of correct answer is statistically significant according to Fisher's exact test ($p < 0.001$).

Table 7.42. Student performance on Task 7.1

		Without *Phasors* tutorial ($N = 665$[a])	With *Phasors* tutorial ($N = 198$[b])
Correct answer (answer E)		26%	39%
Incorrect answers	Answer A	34%	24%
	Answer B	14%	16%
	Answer C	21%	15%
	Answer D	5%	6%

[a]PHYS 123A: SPR 13, SPR 15. PHYS 123C: SPR 14, SPR 16. [b]PHYS 123A: WIN 18.

Although students who had the phasors tutorial performed better, there are still about 60% did not choose correctly. Under both conditions (initial sequence and new sequence of tutorials), answer A was the most common incorrect answer, with the prevalence around 30%. Answers B and C were also somewhat prevalent, ranging from about 15% to 20%.

Since the differences in percentages between different answer choices do not seem vary significantly, we speculate that many students did not know how to approach this problem. In the tutorial *Phasors*, students had seen questions that prompted them to sketch phasors for three slits and then to determine the intensity. This task requires students to use intensity to determine the phasor diagram instead. The results seem to suggest that using intensity to determine the phasor diagram is much more challenging for students than the converse (using phasor diagram to determine intensity).

Table 7.43. Student performance on Task 7.2

		Without *Phasors* tutorial (N = 694[a])	With *Phasors* tutorial (N = 196[b])
Correct answer (answer B)		34%	37%
Incorrect answers	Answer A	34%	36%
	Answer C	8%	7%
	Answer D	9%	5%
	Answer E	16%	14%

[a]PHYS 123C: SPR 13, SPR 15. PHYS 123A: SPR 14, SPR 16. [b]PHYS 123B: WIN 18.

Student performance on Task 7.2 is shown in Table 7.43. The difference in prevalence of correct answer is not statistically significant ($p = 0.349$). The percentages for other answer choices are also very similar between the two conditions.

The results presented in this section suggest that using intensity to determine a phasor diagram is very challenging for many students. Students who had the new sequence of tutorials performed somewhat better (statistically significant) than students who had the initial sequence of tutorials in one of the tasks. The new sequence of tutorials may have some positive impact on student ability to relate intensity to phasor diagram. However, more assessment needs to be done in order to draw a conclusion and it is clear that additional research is required to understand how to improve student performance in such tasks.

7.3 Summary

This chapter discusses the development and assessment of a tutorial on phasors. The development of the tutorial was guided by the research findings reported in chapter 6 that illustrate some of the specific difficulties students have when they use path length difference or phase difference to reason about interference. These include the following: Many students conflate path length difference and phase difference. These students often to treat phase difference as a length quantity, and do not recognize that path length difference is a length quantity, independent of the relative phase of the sources. In addition, many students reason incorrectly about multiple-slit interference when slits are added. Although many can determine the path length difference and seem to be able to superimpose waves in pairs, they are unsuccessful in determining the interference result for more than two waves. (Detailed information on the specific errors that students made are discussed in section 6.4.)

To support students to apply superposition to multiple-slit interference, we developed the tutorial *Phasors*. We believed that phasor diagrams might facilitate student ability to visualize superposition. Furthermore, we thought that it would give students a relatively straightforward process for adding phasors. The tutorial *Phasors* builds on the ideas developed in *Multiple-slit interference* and the other two tutorials on interference as well. It provides students with opportunities to practice determining phase differences and using phasor diagrams.

In order to assess the impact of the new sequence of tutorials on student reasoning about interference, we compare the performance of classes of students who had the new sequence and the performance of classes of students who had the initial sequence. The new sequence of tutorials was tested in two academic quarters. Students who had the new sequence in one of the academic quarters outperformed substantially students who had the initial sequence. Not only a considerably greater portion of students who had the new sequence gave correct answer but also a considerably greater portion of students gave correct reasoning. We have found that a larger percentage of students used the superposition principal after they completed the new sequence of tutorials. Moreover, many students spontaneously used phasor diagrams as part of their explanations and most of them used successfully. However, in the other (later) academic quarter, the improvement was less substantial. In one of the classes, the improvement was statistically significant (although the difference in percentage of correct answer was not as large as in the first quarter), while in the other class it was not statistically significant. We have compared the conditions under which those data were collected for each academic quarter. Several differences have been identified that we speculate may have influenced the results: (1) the lecture instructor in the first quarter discussed above encouraged student to use phasors (although the instructor did not teach phasors in lecture), (2) other related questions given on exams in the later quarter may have influenced student reasoning, (3) the multiple-choice format (no explanation required) used in the later quarter may have triggered more intuitive but incorrect responses.

We have also assessed the effectiveness of the new sequence of tutorials at supporting students in relating intensity to phasor diagrams. We found an improvement in performance (statistically significant) in one class but no improvement (not statistically significant) in another class. The results suggest that after both lecture and tutorial instruction, using intensity to determine phasor diagrams still appears challenging for many students.

To help students differentiate between the path length difference and the phase difference, we modified the tutorial homework *Two-source interference.* We added a problem that involves two contrasting cases: (1) sources that are in phase and (2) sources that are 180^{o} out of phase. This problem provides students an opportunity to compare the values of path length differences in two cases, and to compare the values of phase differences in two cases as well. The exercise is intended to help them recognize that the path length difference is independent on the phase difference between the sources, while the phase difference is dependent. We are planning to assess the effectiveness in the future.

In summary, our preliminary results suggest that the new sequence of tutorials may have some positive impact on student ability to reason about superposition of classical waves. In one of the academic quarters, we found a substantial improvement in student performance. However, the improvement in the other quarter was less significant. The results suggest that additional research is required in order to understand student difficulties in a greater detail so that the instructional strategies can be modified to be less dependent on other aspects of instruction. Further assessment is also needed to ensure that the results are reproducible.

Chapter 8. Conclusion

Superposition is a fundamental principle in many areas of physics, including quantum mechanics and classical wave mechanics. Students are expected to be able to reason about superposition in many different physical scenarios. This dissertation focuses on the teaching and learning of superposition in introductory optics and upper-division quantum mechanics courses. In particular, the research involves three specific areas: (1) student ability to determine quantum probabilities for discrete and continuous cases, (2) student ability to recognize the measurable effects of relative phase in superposition states, and (3) the ability of introductory-level students to apply superposition to the interference of classical waves.

This dissertation illustrates the iterative process through which investigations of student understanding inform the design of curriculum, which is then assessed for its impact on student learning. For each of the content areas above, we present results from an examination of what students do and do not understand after traditional instruction and then, based on the research findings, we discuss the design of curriculum to address the common conceptual and reasoning difficulties that students encounter. Finally, we show how the impact of the curriculum on student understanding is assessed and how the results of assessment have guided us to modify or to revise the curriculum and to conduct further research into student understanding.

To investigate student understanding of superposition, we designed a variety of written tasks to examine in detail different aspects of student thinking. In some cases, we started with tasks that test the most basic level of learning, such as ability to perform computations. We then gradually increased the level of difficulty of the tasks. This allowed us to identify what students do and do not understand at different levels of instruction. In other cases, we began by examining student reasoning ability in relatively complex physical scenarios. To account for the patterns of responses that emerged, we developed hypotheses and designed new tasks to test them. The new tasks often probed student understanding of the prerequisites needed to understand the more complicated scenarios. This procedure allowed us to pinpoint some of the critical barriers that students encounter when reasoning about the relevant physics.

A general result from this step-by-step investigation into student understanding of superposition is that, across different courses and different contexts, students often lack a coherent way to interpret the mathematical formalism used to express superposition. Many students unsuccessfully relate the formalism to the real-world physical phenomena. Often, student understanding of related concepts or ideas are not consistent. In particular, many lack skill in using different representations to reason about specific concepts and ideas.

The specific findings discussed in this dissertation guided the development of curriculum. Several tutorials were developed or revised based on the results. In some cases, the development of curriculum, in turn, drove further investigations of student understanding. Our preliminary results from the various assessments suggest that some of the tutorials appear to have a positive impact on student thinking, although further assessment is needed in some cases to ensure that the results are reproducible. Additional research is also required in order to understand student difficulties in greater detail and to be able to increase the impact on student learning.

Below we summarize some of the specific results. In section 8.1, we summarize the findings from student understanding corresponding to each of the three content areas (see above). In section 8.2, the results from curriculum assessments are summarized. Lastly, section 8.3 discusses possible directions for future research.

8.1 Summary of Results from Investigations of Student Understanding

This section summarizes the results presented in this dissertation. Each of the first three subsections below focuses on one of the three topics (1) determining quantum probabilities for discrete and continuous cases, (2) recognizing the measurable effects of relative phase in superposition states, and (3) applying superposition to classical interference. We then summarize the overall findings and discuss results from curriculum assessments in the last subsection.

8.1.1 Student Ability to Determine Quantum Probabilities for Discrete and Continuous Cases

Chapters 2 and 3 focus on investigating and improving student functional understanding of quantum states and inner products in vector spaces. The overarching goal of the investigation is to improve the ability of students to recognize that probabilities for observables with discrete and/or continuous eigenvalues can be determined using analogous procedures.

We found that students do not always recognize that determining probabilities in quantum mechanics involves taking inner products. Many have difficulty in determining probabilities when the state is not written in terms of the basis states of interest. For example, when given a position-space wave function (not written explicitly in terms of the energy eigenfunctions) and asked for an energy probability, many students fail to take inner products. A further investigation suggested that many students do not recognize that any representation of a quantum state provides information about the probabilities for all observables. They often incorrectly state that only the probability for one observable can be determined; namely the one that corresponds to the basis in which the state is represented. This seemes to explain why students have difficulty in determining probabilities when the quantum state is not expressed in the basis of interest.

We also found that many students do not recognize that a wave function is an inner product. Many confuse a wave function with its associated state vector. This confusion seems to impede student ability to translate expressions between notations and to build a model they can use to determine probabilities for discrete and continuous cases.

Finally, we found that many sophomore-level students lack the fluency of basic computations involving inner products and probabilities. In particular, we identified several common errors students make. Many students treat state vectors like spatial vectors without taking the complex conjugate; many treat a modulus squared of a complex number as a "regular" squared (of a real number) without taking the complex conjugate on tasks that require them to determine probabilities; other common errors involve neglecting to change basis or making basic algebraic errors.

In summary, this research suggests that many students do not have a model that they can use to determine probabilities for discrete and continuous cases in quantum mechanics. Students do not always recognize that determining probabilities involves taking inner products. Moreover, many students do not differentiate between a state vector and its associated wave function. This confusion seems to be widespread and to hinder student ability to determine probabilities.

8.1.2 Student Ability to Recognize the Measurable Effects of Relative Phase in Superposition States

Chapters 4 and 5 center on investigating and improving student ability to recognize the measurable effects of relative phase in superposition states. The results from both spin and infinite square well contexts suggest that many students do not recognize that states that differ only by a relative phase are experimentally distinguishable. Many argue that the probabilities are the same or simply that "the phase does not matter." When prompted to compare the probabilities for a particular observable (that the probabilities are different), many still argue that the probabilities are the same. Additional data from the analogous questions in mathematical context suggest that mathematical proficiency does not seem to be the primary barrier. Rather, students appear to believe that the relative phase does not affect probabilities or modulus squared.

The findings from the tasks that probe student reasoning in quantum and mathematical contexts suggest that student views about the role of relative phases may impact student responses. We conduct a follow-up investigation into student views about the roles of complex numbers in quantum mechanics and electrodynamics. The results suggest that many students do not recognize that complex numbers are used in electromagnetic waves for mathematical convenience. Many believe that the roles complex numbers play in quantum mechanics and electrodynamics are the same.

8.1.3 Student Ability to Apply Superposition to Interference of Classical Waves

The results from the first two parts of the investigation led us to question the extent to which students in introductory physics courses understand superposition. Chapters 6 presents an investigation into student ability to apply superposition to the interference of classical waves in an introductory-level waves and optics course. Chapter 7 discusses the development and assessment of tutorials that are intended to facilitate student reasoning about superposition.

We found that many students do not differentiate between phase difference and path length difference. Some express phase difference in terms of wavelength. In addition, many students do not treat path length difference and source separation as purely length quantities. Many "add" the phase difference between the sources to the path length difference (or to the source separation). The findings suggest that many students conflate the concepts of phase difference and path length difference.

When students reason about superposition, many are able to determine the path length difference and to superimpose waves in pairs, but unsuccessfully determine the interference result for more than two sources or slits. For example, many students tend to reason that if a pair of waves are in phase and another pair of waves are in phase, then all the waves must be in phase. This type of reasoning is not always correct since one pair of waves can be completely out of phase with another pair of waves, resulting into a point of zero intensity. Other students tend to reason that if some pairings of waves have maximum constructive interference (e.g., pair 1 and 2 and pair 3 and 4), but some pairings have completely destructive interference (e.g., pair 1 and 3 and pair 2 and 4), then the net result is neither a principal maximum or minimum. These students do not seem to recognize that the cancellation of the two pairs of the waves individually results in the complete cancellation of all four waves. Although not very common, some students tend to add the path length differences between each pairing of waves and then use the *total* path length difference to determine the interference pattern. It seems that these students attempt to determine the net effect of superposition using an incorrect strategy.

In summary, the results discussed in chapter 6 suggest that the concept of phase difference is abstract and challenging for many students. It is common for students fail to differentiate between phase difference and path length difference. Although the tutorials on physical optics have been well tested [40], many have difficulty with using phase difference or path length difference to reason about superposition even after tutorial instruction.

8.1.4 Overall Findings

The research into the three areas above provide us with insights from different perspectives into how students reason about superposition. The findings from the first area suggest that when given the result of superposition, many students do not recognize that the components can be determined using inner products. The second and third areas show how students relate the relative phase (or phase difference) between the components to the result of superposition. We found that many quantum mechanics students do not recognize that the relative phase can affect the result of superposition; the introductory-level students often lack skill in using phase difference successfully to determine reason about superposition.

The results discussed in this dissertation suggest that across different courses and different contexts, students often lack a coherent way to interpret the mathematical formalism related to superposition in various contexts. Many students unsuccessfully relate the formalism to the real-world physical phenomena.

8.2 Summary of Results from Curriculum Assessments

The research findings discussed in the previous section drove the design of curriculum in both introductory and quantum physics courses. The results from various assessments suggest that some of the new and modified tutorials seem to be effective at supporting student reasoning about superposition, while others are less successful and require additional research. Below we briefly summarize the assessment results for each of the tutorials.

8.2.1.a Tutorial *Probability Amplitude*

The new tutorial *Probability amplitude* was intended to help students develop a functional understanding of quantum states and inner products. This tutorial seems to have a positive impact on student ability to translate inner products from Dirac to wave function notations and on student ability to determine position probabilities. On the other hand, the difficulty in determining energy probabilities when given position-space wave functions (not written explicitly in terms of energy eigenfunctions) appears to persist even after both lecture and tutorial instruction. However, there were some complications for the implementation of tutorials that may have influenced the results. Additional assessments and further investigations are needed.

8.2.1.b Tutorial *Quantum Interference with Spin States*

The new tutorial *Quantum interference with spin states* was intended to help students recognize that the relative phase between the eigenstates in a superposition states are needed to account for the results of Stern-Gerlach experiments. The results from the assessment of the first version revealed the extent to which the ideas developed in this tutorial are challenging for many sophomore-level students. This finding drove a further investigation into student understanding. Our results show that many students do not spontaneously use complex numbers for coefficients. They do not recognize that the coefficients can have different phases when asked to write possible states to describe a particular set of statistical results from measuring one observable. This research finding provides insights into potential instructional strategies to improve the tutorial that need to be tested in the future.

8.2.1.c Tutorial *Superposition in Quantum Mechanics*

The revised version of the tutorial *Superposition in quantum mechanics* was focused on improving student ability to recognize the measurable effect of relative phase. The assessment suggests that the tutorial is effective. After tutorial instruction, many students explain correctly that quantum states that differ only by a relative phase between the energy eigenstates can be distinguished by measuring position.

8.2.1.d Tutorial *Phasors*

The new tutorial *Phasors* was intended to help students make use of phasor diagrams to reason about superposition. Our preliminary results from the assessments suggest that the addition of the phasors tutorial to the existing sequence of tutorials on interference may have some positive impact on student ability to reason about superposition of classical waves. In one of the academic quarters, we found a substantial improvement in student performance and reasoning. However, the improvement in a second quarter was less significant. The results suggest that additional research is required in order to understand student difficulties in a greater detail so that the instructional strategies can be modified to be less dependent on other aspects of instruction. Further assessment is also needed to ensure that the results are reproducible.

8.3 Future Research

The research presented in this dissertation provides insights into how students reason about superposition in several interrelated content areas. The specific findings drove curriculum development and assessment. Our results from assessments suggest that the tutorials developed had varying degrees of success. Additional research is needed to increase the impact of the tutorials on student reasoning about superposition. In this section, we discuss possible avenues for future research.

The results from the assessment of the new tutorial *Probability amplitude* and the modified version of *Representation of wave functions* suggest that after (both lecture and) tutorial instruction, many students still do not recognize that determining probability for both discrete and continuous cases involve taking inner products. When these tutorials were administered, they were not given as a sequence. Instead, students had a sequence of tutorials on time dependence after they finished *Representation of wave functions* and before they had *Probability amplitude.* This decision was made in order to have tutorials better synchronize with lecture. We hypothesize that these tutorials might be more effective if they are given to students as a sequence since the tutorial *Probability amplitude* builds on the ideas developed in tutorial *Representation of wave functions*. We are planning to assess this sequence of tutorials in the future.

As discussed in chapter 5, the lack of positive results from the assessment of the tutorial *Quantum interference with spin states* motivated further investigation into student learning. We found that many students do not recognize that the coefficients can have different phases when asked to write possible states to describe a particular set of statistical results from measuring one observable. We believe that this difficulty needs to be more explicitly addressed in order for students to develop the ideas in the tutorial *Quantum interference with spin states.*

Relative phase is one of the underlying concept of time dependence in quantum mechanics. We have started to modify the tutorial *Time dependence in quantum mechanics* to make clear to students that the relative phases between the energy eigenstates change in time, resulting in time-dependent probability density in position space. In particular, this tutorial makes use of phasors to help students visualize the relative phase. It is worthwhile to mention that the phasor representation is also used in the tutorial *Two-state time dependence* intended for sophomore-level students. Additional research could investigate how phasor representation facilitate (or hinder) student reasoning about relative phase and time dependence.

The tutorial *Phasors* used in the introductory wave and optics course appears to have a positive impact. However, the improvements vary over different academic quarters. Additional assessments are needed. Further research could probe in detail student ability to use phasor diagrams. A longitudinal study could be conducted to examine how student facility with phasor representation evolves from introductory courses to upper-division quantum mechanics courses.

General summary: Overall the results of this research indicate how difficult it can be for students to develop a functional understanding of superposition. Although the basic concept may appear simple to instructors, students can struggle in applying it in a wide variety of situations. A general finding is that, from introductory to upper-division quantum physics courses, students often lack a coherent way to interpret the mathematical formalism used to express superposition. Many students unsuccessfully relate the mathematical formalism to the real-world physical phenomena. Although the research focuses on a specific topic, superposition, this finding may be generalized to student learning of other topics as well. We argue that instructions and research should put efforts into explicitly supporting students to relate formalism to real-world phenomena throughout all the topics that students are taught in physics courses. We believe that this is necessary since our (and other) results show that students do not develop conceptual understanding and reasoning skills at once but rather in stages. The research discussed in this dissertation serves to demonstrate (as has other work by the UW PEG and the physics education research community) the need for an iterative cycle of research and curriculum development that focuses on improving student ability to relate the formalism and real-world effects.

BIBLIOGRAPHY

[1] C. Singh, Student understanding of quantum mechanics, Am. J. Phys. **69**, 885 (2001).

[2] C. Singh, Student understanding of quantum mechanics at the beginning of graduate instruction, Am. J. Phys. **76**, 277 (2008).

[3] G. Zhu and C. Singh, Surveying students' understanding of quantum mechanics in one spatial dimension, Am. J. Phys. **80**, 252 (2012).

[4] G. Zhu and C. Singh, Improving students understanding of quantum measurement. I. Investigation of difficulties, Phys. Rev. ST Phys. Educ. Res. 8, 010117 (2012).

[5] E. Cataloglu and R. W. Robinett, Testing the development of student conceptual and visualization understanding in quantum mechanics through the undergraduate career, Am. J. Phys. **70**, 238 (2002).

[6] P. J. Emigh, G. Passante, and P. S. Shaffer, Student understanding of time dependence in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020112 (2015).

[7] G. Passante, P. J. Emigh, and P. S. Shaffer, Examining student ideas about energy measurements on quantum states across undergraduate and graduate levels, Phys. Rev. ST Phys. Educ. Res. **11**, 020111 (2015).

[8] C. Singh and E. Marshman, Review of student difficulties in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020117 (2015).

[9] E. Marshman and C. Singh, Framework for understanding student difficulties in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020119 (2015).

[10] G. Zhu and C. Singh, Improving students' understanding of quantum mechanics via the Stern-Gerlach experiment, Am. J. Phys. 79, 499 (2011).

[11] G. Passante, P. J. Emigh, and P. S. Shaffer, Student ability to distinguish between superposition states and mixed states in quantum mechanics, Phys. Rev. ST Phys. Educ. Res. **11**, 020135 (2015).

[12] A. Kohnle, C. Baily, and S. Ruby, Investigating the influence of visualization on student understanding of quantum superposition, presented at the *Physics Education Research Conference 2014*, Minneapolis, MN (2014).

[13] E. Marshman and C. Singh, Investigating and improving student understanding of quantum mechanics in the context of single photon interference, Phys. Rev. Phys. Educ. Res. **13**, 010117 (2017).

[14] R. Sayer, A. Maries, and C. Singh, Quantum interactive learning tutorial on the double-slit experiment to improve student understanding of quantum mechanics, Phys. Rev. Phys. Educ. Res. **13**, 010123 (2017).

[15] M. Michelini and G. Zuccarini, University students' reasoning on physical information encoded in quantum state at a point in time, presented at the *Physics Education Research Conference 2014*, Minneapolis, MN (2014).

[16] H. G. Close, C. C. Schiber, E. W. Close, and D. Donnelly, Students' dynamic geometric reasoning about quantum spin-1/2 states, presented at the *Physics Education Research Conference 2013*, Portland, OR (2013).

[17] T. Wan, P. J. Emigh, G. Passante, and P. S. Shaffer, Student understanding of period in introductory and quantum physics courses, presented at the *Physics Education Research Conference 2016*, Sacramento, CA (2016).

[18] T. Wan, P. J. Emigh, and P. S. Shaffer, Student understanding of the measurable effects of relative phases in superposition states, presented at the *Physics Education Research Conference 2017*, Cincinnati, OH (2017).

[19] A. Kohnle and G. Passante, Characterizing representational learning: A combined simulation and tutorial on perturbation theory, Phys. Rev. Phys. Educ. Res. **13**, 020131 (2017).

[20] C. Singh and E. Marshman, Investigating Student Difficulties with Dirac Notation, presented at the *Physics Education Research Conference 2013*, Portland, OR (2013).

[21] M. Wittmann, J. Morgan, and L. Bao, Addressing student models of energy loss in quantum tunneling, Eur. J. Phys. 26, 939 (2005).

[22] E. Marshman and C. Singh, Investigating and improving student understanding of the probability distributions for measuring physical observables in quantum mechanics, Eur. J. Phys. 38 (2), 025705.

[23] E. Marshman and C. Singh, Student difficulties with quantum states while translating state vectors in Dirac notation to wave functions in position and momentum representations, presented at the *Physics Education Research Conference 2015*, College Park, MD (2015).

[24] H. R. Sadaghiani and S. J. Pollock, Quantum mechanics concept assessment: Development and validation study, Phys. Rev. Phys. Educ. Res. **11**, 010110 (2015).

[25] G. Passante, P. J. Emigh, T. Wan, and P. S. Shaffer, Investigating student understanding of perturbation theory and inner products of functions, presented at the *Physics Education Research Conference 2015*, College Park, MD (2015).

[26] E. Marshman and C. Singh, Investigating student difficulties with time dependence of expectation values in quantum mechanics, presented at the *Physics Education Research Conference 2013*, Portland, OR (2013).

[27] S. B. McKagan, K. K. Perkins, and C. E. Wieman, Design and validation of Quantum Mechanics Conceptual Survey, Phys. Rev. ST Phys. Educ. Res. **6**, 020121 (2010).

[28] J. T. Morgan and M. C. Wittmann, *Examining the evolution of student ideas about quantum tunneling*, AIP Conf. Proc. **818**, 73 (2006).

[29] C. Singh, Student Difficulties with Quantum Mechanics Formalism, presented at the *Physics Education Research Conference 2006*, Syracuse, New York (2006).

[30] B. W. Dreyfus, A. Elby, A. Gupta, and E. R. Sohr, Mathematical sense-making in quantum mechanics: An initial peek, Phys. Rev. Phys. Educ. Res. **13**, 020141 (2017).

[31] V. Dini and D. Hammer, Case study of a successful learner's epistemological framings of quantum mechanics, Phys. Rev. Phys. Educ. Res. **13**, 010124 (2017).

[32] B. Modir, J. D. Thompson, and E. C. Sayre, Students' epistemological framing in quantum mechanics problem solving, Phys. Rev. Phys. Educ. Res. **13**, 020108 (2017).

[33] E. Gire and E. Price, Structural features of algebraic quantum notations, Phys. Rev. ST Phys. Educ. Res. **11**, 020109 (2015).

[34] A. Mason and C. Singh, Do advanced students learn from their mistakes without explicit intervention?, Am. J. Phys. **78**, 760 (2010).

[35] S. Y. Lin and C. Singh, Categorization of quantum mechanics problems by professors and students, Eur. J. Phys. 31, 57 (2010).

[36] E. C. Sayre and A. F. Heckler, Peaks and decays of student knowledge in an introductory E&M course, Phys. Rev. ST Phys. Educ. Res. **5**, 013101 (2009).

[37] M. E. Bouton, Context, time, and memory retrieval in the interference paradigms of Pavlovian learning, Psychol. Bull. **114**, 80 (1993).

[38] B. S. Ambrose, P. S. Shaffer, R. N. Steinberg, and L. C. McDermott, An investigation of student understanding of two-source interference and single-slit diffraction, Am. J. Phys. **67**, 146-155 (1999).

[39] S. Vokos, P. S. Shaffer, B. S. Ambrose, and L. C. McDermott, Student understanding of the wave nature of matter: Diffraction and interference of particles, Am. J. Phys. **67**, S5 (1999).

[40] K. Wosilait, P. R. L. Heron, P. S. Shaffer, and L. C. McDermott, Addressing student difficulties in applying a wave model to the interference and diffraction of light, Am. J. Phys. **67**, S5 (1999).

[41] P. Barniol and G. Zavala, Force, velocity, and work: The effects of different contexts on students' understanding of vector concepts using isomorphic problems, Phys. Rev. ST Phys. Educ. Res. **10**, 020115 (2014).

[42] S. Rainson, G. Transtromer, and L. Viennot, Students understanding of superposition of electric fields, Am. J. Phys. **62**, 1026 (1994).

[43] C. Singh, Students understanding of symmetry and Gauss's law of electricity, Am. J. Phys. **74**, 923 (2006).

[44] Jing Li and C. Singh, Investigating and improving introductory physics students' understanding of the electric field and superposition principle, European Journal of Physics **38,** 055702 (2017).

[45] Jing Li and C. Singh, Investigating and improving introductory physics students' understanding of symmetry and Gauss's law, European Journal of Physics **39**, 015702 (2017).

[46] http://phet.colorado.edu.

[47] C. Singh, "Interactive Learning Tutorials on Quantum Mechanics," Am. J. Phys. **80**, 148 (2012).

[48] L. C. McDermott, P. S. Shaffer, and M. L. Rosenquiest, *Physics By Inquiry: An Introduction to Physics and the Physical Sciences* (Wiley, New York, 1996)

[49] L. C. McDermott, P. S. Shaffer, and the Physics Education Group, *Tutorials in Introductory Physics,* Prentice Hall College Div (2001).

[50] D. Hestenes, M. Wells, and G. Swackhamer, Force Concept Inventory, Phys. Teach. **30**, 141 (1992).

[51] University of Washington Office of Admissions About UW, https://admit.washington.edu/why-uw/about.

[52] P. A. Tipler and G. Mosca, *Physics for scientists and engineers*, Macmillan, 2007.

[53] E. Mazur, C. H. Crouch, D. Pedigo, P. A. Dournashkin, and R. J. Bieniek, *Principles and practice of physics*, Pearson, 2015.

[54] D. H. McIntyre, *Quantum Mechanics: A Paradigm Approach,* 1st edition (2012).

[55] D. J. Griffiths, *Introduction to Quantum Mechanics*, Pearson Prentice Hall (2005).

[56] L. C. McDermott, Oersted Medal Lecture 2001: "Physics Education Research--The key to student learning", Am. J. Phys. **69**, 1127 (2001).

[57] L. C. McDermott, Millikan Lecture 1990: What we teach and what is learned--Closing the gap, Am. J. Phys. **59**, 301 (1991).

[58] P. S. Shaffer and L. C. McDermott, Research as a guide for curriculum development: An example from introductory electricity. Part II: Design of instructional strategies, Am. J. Phys. **60**, 1003 (1992).

[59] K. Krippendorff, *Content Analysis: An Introduction to Its Methodology* (Sage, Thousand Oaks, 2013).

[60] www.biostathandbook.com/fishers.html

[61] www.biostathandbook.com/chiind.html

[62] P. R. L. Heron, Effect of lecture instruction on student performance on qualitative questions, Phys. Rev. ST Phys. Educ. Res. **11**, 010102 (2015).

[63] Q. McNemar, Note on the sampling error of the difference between correlated proportions or percentages, Psychometrika, **12** (2) 153-157 (1947).

[64] S. Goldhaber, S. J. Pollock, M. Dubson, P. Beale, and K. K. Perkins, Transforming Upper-Division Quantum Mechanics: Learning Goals and Assessment, presented at the Physics Education Research Conference 2009, Ann Arbor, Michigan, 2009.

[65] http://www.colorado.edu/sei/documents/phys3220-learning-goals.pdf

[66] P. J. Emigh, Ph.D. thesis, University of Washington, 2015

[67] A. F. Heckler and T. M. Scaife, Adding and subtracting vectors: The problem with the arrow representation, Phys. Rev. ST Phys. Educ. Res. **11**, 010101 (2015).

[68] E. M. Smith, J. P. Zwolak, and C. A. Manogue, Student difficulties with complex numbers, presented at the *Physics Education Research Conference 2015*, College Park, MD (2015).

[69] G. Passante, Energy measurement resources in spins-first and position-first quantum mechanics, presented at the *Physics Education Research Conference 2016*, Sacramento, CA (2016).

[70] P. J. Emigh, G. Passante, and P.S. Shaffer, Student understanding of superposition: Vectors and wave functions, presented at the *Physics Education Research Conference 2016*, Sacramento, CA (2016).

[71] J. Day, H. Nakahara, and D. Bonn, Teaching standard deviation by building from student invention, The Physics Teacher, 48, 546-548 (2010).

[72] C. C. Chase, J. T. Shemwell, and D. L. Schwartz, Explaining across contrasting cases from deep understanding in science: An example using interactive simulations, Proceedings of the 9th international conference of the learning sciences (2010).

[73] M. C. Wittmann, Ph.D. thesis, University of Maryland, 1998.

[74] V. Mesic, E. Hajder, K. Neumann, and N. Erceg, Comparing different approaches to visulizing light waves: An experimental study on teaching wave optics, Phys. Rev. Phys. Educ. Res. **12**, 010135 (2016).

[75] L. Maurines, Geometrical reasoning in wave situations: The case of light diffraction and coherent illumination optical imaging, Int. J. Sci. Educ. **32**, 1895 (2010).

[76] S. K. Sengoren, How do Turkish high school graduates use the wave theory of light to explain optics phenomena?, Phys. Educ. **45**, 253 (2010).

[77] K. Wosilait, Ph.D. thesis, University of Washington, 1996.

[78] https://www.researchgate.net/publication/326069616_Engaging_reflective_thinking_during_exam-like_situations_Slowing_students_down_on_short-answer_questions_increases_perfromance

[79] A. Crouse, Ph.D. thesis, University of Washington, 2007.

APPENDIX A

Tutorial In-class Worksheets

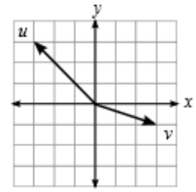
This appendix consists of copies of the tutorial in-class worksheets discussed in this dissertation.

DIRAC NOTATION

QM 1

I. Inner products for spatial vectors

Consider the vectors $\vec{u}$ and $\vec{v}$ shown at right.



A. Write mathematical expressions for these vectors in terms of the basis vectors $\hat{x}$ and $\hat{y}$.

B. Evaluate the inner product $\vec{u} \cdot \vec{v}$. Explain why the inner product results in a *scalar* rather than a vector.

C. In the space below, sketch a pair of vectors that have an inner product with the indicated sign. In each case, explain how you determined your answer.

1. Positive

2. Negative

3. Zero

D. Consider two vectors, $\vec{a}$ and $\vec{b}$, each of which has n components instead of two.

1. Use the symbol Σ to write expressions for $\vec{a}$ and $\vec{b}$ in terms of a set of n orthonormal basis vectors, $\hat{x}_i$ (where i = 1, 2, …, n).

2. Determine the value of the inner product between two basis vectors: $\hat{x}_i \cdot \hat{x}_j$. (*Hint:* Consider the cases where $i = j$ and $i \neq j$ separately.)

3. Use the symbol Σ to write a general expression for $\vec{a} \cdot \vec{b}$. Explain why only one sum is necessary in this expression.

✓ Check your results with an instructor.

Figure A.1: *Dirac notation* in-class tutorial worksheet administered in Winter 2016 to PHYS 225 (five pages)

II. Inner products in Dirac notation

Consider a 6-dimensional Hilbert space with orthonormal basis state vectors $|\varphi_i\rangle$, where *i* is an integer between 1 and 6. Recall that the inner product between two state vectors, $|\alpha\rangle$ and $|\beta\rangle$, written in Dirac notation is given by the expression $\langle\alpha|\beta\rangle$.

A. Determine the value of the inner product between two basis state vectors: $\langle\varphi_i|\varphi_j\rangle$.

How is your answer similar to the inner product between two spatial basis vectors, $\hat{x}_i \cdot \hat{x}_j$? Explain.

B. Consider the following state vector for this system: $|\psi_A\rangle = 2|\varphi_1\rangle - |\varphi_2\rangle + i|\varphi_4\rangle + |\varphi_6\rangle$.

1. Write the dual vector, $\langle\psi_A|$, corresponding to the state vector above. Explain how you arrived at your answer.

2. Determine the inner product between this state vector and itself, $\langle\psi_A|\psi_A\rangle$.

3. How is the inner product between a state vector and itself related to the *magnitude* of that state vector? Explain.

4. How is the process for finding the inner product between two state vectors in Dirac notation different from the process for finding the inner product between two spatial vectors? Explain.

Consider a different state vector for the same system: $|\psi_B\rangle = i|\varphi_1\rangle - 2|\varphi_3\rangle + i|\varphi_4\rangle + |\varphi_6\rangle$.

C. Is the magnitude of $|\psi_B\rangle$ *greater than*, *less than*, or *equal to* the magnitude of $|\psi_A\rangle$? Explain.

D. Consider the dialogue between two students below.

Student 1: "Terms that are imaginary don't contribute at all to the magnitude of a state vector, so the magnitude of state B is only 5, which is smaller than the magnitude of state A."

Student 2: "You're right that B is smaller than A, but I think the imaginary terms take away from the magnitude of the state, since $i^2 = -1$. So the magnitude of B is really only 3."

Both students calculated the magnitude of $|\psi_B\rangle$ incorrectly. Identify the error(s) made by each student.

E. Determine the value of the following inner product: $\langle\psi_A|\psi_B\rangle$.

F. Do you expect the inner product $\langle\psi_B|\psi_A\rangle$ to be equal to $\langle\psi_A|\psi_B\rangle$? Explain your reasoning.

How does your answer differ from what you would expect for the inner product between two spatial vectors? Explain.

✓ Check your results with an instructor

III. Changes of basis

Recall that the state vector for a spin-1/2 particle (such as an electron) may be written in terms of the two orthonormal basis states corresponding to the *z*-direction: $|+\rangle_z$ and $|-\rangle_z$. Such a state vector may also be written in terms of the two orthonormal basis states corresponding to the *x*-direction: $|+\rangle_x$ and $|-\rangle_x$.

Suppose an electron is described by the state vector $|\chi\rangle = \frac{4}{5}|+\rangle_z + \frac{3}{5}|-\rangle_z$.

A. Calculate the inner product ${}_z\langle+|\chi\rangle$. Clearly show each step in your calculation, including any use of orthonormality.

B. Suppose you are interested in the inner product ${}_x\langle+|\chi\rangle$. What additional information is necessary in order to calculate this inner product? Explain.

C. Consider the dialogue below between three students who are attempting to determine the inner product ${}_x\langle+|\chi\rangle$.

Student 1: "This is just like the first inner product: ${}_x\langle+|-\rangle_z$ is zero because the plus and minus states are always orthogonal to each other. Therefore, the answer is 4/5."

Student 2: "The *x*- and z-directions are orthogonal to each other, so $|+\rangle_x$ is orthogonal to both $|+\rangle_z$ and $|-\rangle_z$. That means the answer is zero."

Student 3: "I don't think this inner product has a well-defined answer. The two state vectors are written in terms of different bases, so we can't find the inner product between them."

With which student(s), if any do you agree? Explain your reasoning.

D. Determine the inner product ${}_x\langle+|\chi\rangle$. Clearly show each step in your calculation.

E. Consider another electron described by the state vector $|\psi\rangle = \frac{1}{2}|+\rangle_x + \frac{\sqrt{3}}{2}|-\rangle_x$.

1. Calculate each of the following inner products.

   ${}_z\langle +|\psi\rangle$

   ${}_x\langle +|\psi\rangle$

2. Which of the two inner products above is easier to calculate? Explain.

3. Is it possible to rewrite $|\psi\rangle$ in terms of the basis state vectors $|+\rangle_z$ and $|-\rangle_z$? Explain why or why not.

   If it is possible, write $|\psi\rangle$ in terms of these state vectors. Clearly show your work.

F. Explain why it is important to be able to rewrite a state vector in a different basis. When is it most necessary to change from one basis to another?

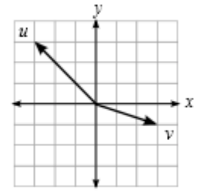
✓ Check your results with an instructor.

DIRAC NOTATION

QM
1

I. Inner products for spatial vectors

Consider the vectors $\vec{u}$ and $\vec{v}$ shown at right.



A. Write mathematical expressions for these vectors in terms of the basis vectors $\hat{x}$ and $\hat{y}$.

B. Evaluate the inner product $\vec{u}\cdot\vec{v}$. Does the inner product result in a *scalar* or a *vector*? Explain.

C. In the space below, sketch a pair of vectors that have an inner product with the indicated sign. In each case, explain how you determined your answer.

1. Positive

2. Negative

3. Zero

✓ Check your results with an instructor.

II. Inner products in Dirac notation

Recall that the state vector for a spin-1/2 particle (such as an electron) may be written in terms of the two orthonormal basis states corresponding to the *z*-direction: $\left|+\right\rangle_z$ and $\left|-\right\rangle_z$. The inner product between two state vectors, $\left|\alpha\right\rangle$ and $\left|\beta\right\rangle$, written in Dirac notation is given by the expression $\left\langle\alpha\middle|\beta\right\rangle$.

A. Determine the value of the inner product between two basis state vectors:

${}_z\langle+|+\rangle_z$ ${}_z\langle-|+\rangle_z$

${}_z\langle+|-\rangle_z$ ${}_z\langle-|-\rangle_z$

How is your answer analogous to the inner product between two spatial basis vectors? Explain.

Figure A.2: *Dirac notation* in-class tutorial worksheet administered in Winter 2017 to PHYS 225 (five pages).

QM 2 ***Dirac notation***

B. Suppose an electron is described by the state vector $|\chi\rangle = N\left(4i|+\rangle_z + 3|-\rangle_z\right)$, where N is a positive real number. (Recall that the coefficients for a state vector can be complex.)

1. Write the dual vector, $\langle\chi|$, corresponding to the state vector above.

2. Determine the inner product between this state vector and itself, $\langle\chi|\chi\rangle$.

3. How is the inner product between a *spatial* vector and itself *(i.e.,* $\vec{u}\cdot\vec{u}$ *)* related to the *magnitude* of that spatial vector?

   How is the inner product between a *state* vector and itself (*i.e.*, $\langle\chi|\chi\rangle$) related to the magnitude of that state vector? Explain.

   Can the inner product between a state vector and itself be negative? Can it be imaginary? Explain.

   Resolve any inconsistencies with your answers to question 2.

4. How is the process for finding the inner product between two state vectors in Dirac notation different from the process for finding the inner product between two spatial vectors? Explain.

C. Consider the same state vector $|\chi\rangle = N\left(4i|+\rangle_z + 3|-\rangle_z\right)$.

1. Calculate the inner product ${}_z\langle+|\chi\rangle$. Clearly show each step in your calculation, including any use of orthonormality.

2. Suppose this electron is sent through a Stern-Gerlach apparatus. Determine the probability that spin up in the *z*-direction would be measured in terms of *N*. Show your work.

3. Determine the probability that spin down in the *z*-direction would be measured in terms of *N*. Show your work.

4. Determine the constant *N*. (*Hint:* What is the probability that either spin up or spin down in the *z*-direction would be measured?)

5. The constant *N* is usually called the normalization constant. Why is a state vector usually normalized? Explain.

Consider a different state vector for another electron: $|\beta\rangle = N'\left(3|+\rangle_z - 4i|-\rangle_z\right)$.

D. Determine the value of the following inner products:

$\langle\chi|\beta\rangle$

$\langle\beta|\chi\rangle$

E. Explain why the two inner products have different values.

Do you expect the inner products have different values if they were spatial vectors? Explain.

✓ Check your results with an instructor.

III. Changes of basis

Suppose an electron is described by the state vector $|\psi\rangle = \frac{1}{2}|+\rangle_z + \frac{\sqrt{3}}{2}|-\rangle_z$.

A. Calculate the inner product ${}_z\langle +|\psi\rangle$.

B. Is this state vector normalized? Explain by discussing the probabilities of measuring spin in the z-direction.

C. *Predict* whether the magnitude of the inner product ${}_x\langle +|\psi\rangle$ would be *greater than*, *less than*, or *equal to* the magnitude of the inner product ${}_z\langle +|\psi\rangle$. Explain.

D. Consider the dialogue below between three students who are attempting to determine the inner product ${}_x\langle +|\psi\rangle$.

Student 1: "This is just like the first inner product: ${}_x\langle +|-\rangle_z$ is zero because the plus and minus states are always orthogonal to each other. Therefore, ${}_x\langle +|\psi\rangle$ is 1/2."

Student 2: "The *x*- and z-directions are orthogonal to each other, so $|+\rangle_x$ is orthogonal to both $|+\rangle_z$ and $|-\rangle_z$. That means the answer is zero."

Student 3: "I don't think the inner product of is ${}_x\langle +|+\rangle_z$ zero. If we send an electron prepared in state $|+\rangle_z$ to a S-G apparatus with magnetic field in x-direction, we would have 50/50 chance of measuring spin up and spin down. Thus the inner product can't be zero."

With which student(s), if any do you agree? Explain your reasoning.

E. Determine the inner product ${}_x\langle + | \psi \rangle$. Clearly show each step in your calculation.

Resolve any inconsistencies with your answers to question C.

F. Is it possible to rewrite $|\psi\rangle$ in terms of the basis state vectors $|+\rangle_x$ and $|-\rangle_x$? Explain why or why not.

If it is possible, do you expect the coefficients in terms of $|+\rangle_x$ and $|-\rangle_x$ to be the same as the coefficients in terms of $|+\rangle_z$ and $|-\rangle_z$? Explain.

Write $|\psi\rangle$ in terms of these state vectors. Clearly show your work.

✓ Check your results with an instructor.

TREATING FUNCTIONS AS VECTORS

QM
1

I. Vectors

Consider the vector given by the expression $\vec{v} = 3\hat{x} - 2\hat{y}$.

A. What do $\hat{x}$ and $\hat{y}$ represent in the expression above? How are they related to each other?

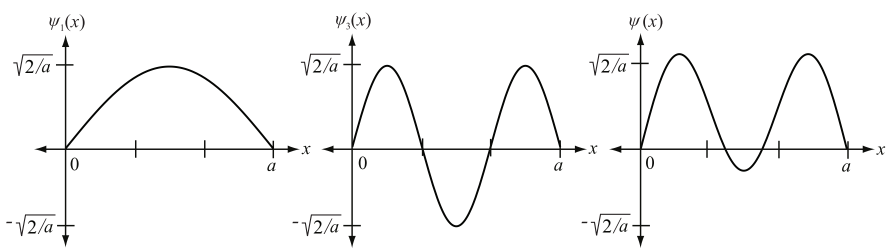
B. Explain why the inner (dot) product $\hat{x} \cdot \vec{v}$ is equal to the x-component of the vector $\vec{v}$.

C. Write $\vec{v}$ as a column vector; express each element as both an inner product and a number.

II. Position space

Consider the three functions defined from $x = 0$ to $x = a$ below.

$$\psi_1(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{\pi x}{a}\right) \qquad \psi_3(x) = \sqrt{\frac{2}{a}} \sin\left(\frac{3\pi x}{a}\right) \qquad \psi(x) = \frac{1}{\sqrt{13}}\left(2\psi_1(x) + 3\psi_3(x)\right)$$



A. Divide each graph into three regions of equal width. Approximate the value of each function by its value at the center of the region and write your results in the table at right.

B. In the space below, sketch a new graph of each function by assuming it is equal to the value written in the table throughout each region (*i.e.,* each graph should be a step function with three levels).

x	$\psi_1(x)$	$\psi_3(x)$	$\psi(x)$
x_1			
x_2			
x_3			

Tutorials in Physics: Quantum Mechanics
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Figure A.3: *Treating Functions as Vectors* in-class tutorial worksheet administered in Autumn 2015 to PHYS 324 (four pages).

C. Represent each function as a column vector in which the entries are equal to the values of the function in each of the equal-width regions.

D. Describe the basis vectors for the column vector representation from the previous question.

In the space below, sketch graphs of functions (from $x = 0$ to $x = a$) to represent the basis vectors, and write the corresponding column vectors next to each sketch.

E. Describe how the column vector representation used above could be modified to provide a *better* approximation to a function.

F. Assume you were to divide your graphs into N equally-sized regions.

1. As N becomes very large, how do the column vectors that approximate each function change?

   What happens to the number of basis states and to their graphs?

One way of describing a function is as a list of numbers: a single number, $\psi(x)$, associated with each value of position, x.

2. How is the column vector representation of the functions on this page similar to this definition of a function? Explain.

✓ Check your results with an instructor.

III. Inner products

A. Consider two vectors given by $\vec{u} = 2\hat{x} + 3\hat{y}$ and $\vec{v} = 3\hat{x} - 2\hat{y}$.

1. Write each of these vectors as a column vector.

2. Use the column vector representation to compute the dot product $\vec{u} \cdot \vec{v}$.

   Use your result to explain how $\vec{u}$ and $\vec{v}$ are related.

B. Consider the function given by the expression $\psi_1(x)\psi_3(x)$.

1. Use the approximation from section II to represent this function in the table at right.

2. Sketch a graph of this approximation in the space below.

x	$\psi_1(x)\psi_3(x)$
x_1	
x_2	
x_3	

3. Determine an approximate value for the integral $\int_0^a \psi_1(x)\psi_3(x)\,dx$. Describe each step of the procedure you use.

   How did you approximate the infinitesimal width dx in your procedure?

The inner product between two *functions,* $b(x)$ and $c(x)$, is defined to be the following integral: $\int b^*(x)c(x)\,dx$, where $b^*(x)$ indicates the complex conjugate of $b(x)$.

C. Explain why the expression $b^*(x)c(x)$, without an integral, does *not* give the inner product of two functions.

D. Describe how to approximate the inner product between two functions of *x* using your result from part B.

How does this procedure for finding the inner product differ from the procedure you used in question A2?

What does your result for the inner product of $\psi_1(x)$ and $\psi_3(x)$ suggest about how these two functions are related to each other?

E. Determine an approximate value for the inner product between $\psi_1(x)$ and $\psi(x)$.

Compare your answer to the original definition of $\psi(x)$ in terms of $\psi_1(x)$ and $\psi_3(x)$.

F. Should the inner product between $\psi_1(x)$ and itself be *greater than, less than,* or *equal to* one? Explain.

REPRESENTATIONS OF WAVE FUNCTIONS

QM 1

I. The energy basis

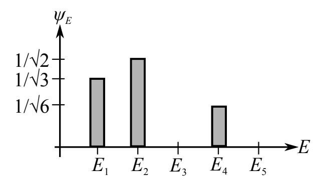
Consider a particle in the quantum mechanical infinite square well of width a described by

$|\psi\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, where $|\psi_n\rangle$ is the n^{th} energy eigenstate with eigenvalue E_n.

A. List the possible results of an energy measurement on the particle described by the state above and the probability associated with each result. Explain how you determined your answers from the given representation of this quantum state.

B. The quantum state above is said to be represented in the *energy basis*. Explain why this name is appropriate for this representation.

C. The graph shown at right is a visual representation of the state given above.



1. Would it still be appropriate to say that the state is represented in the energy basis? Explain.

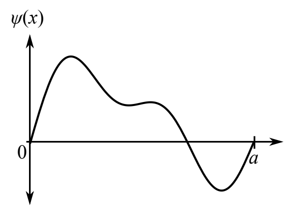
2. Give a physical interpretation for the vertical component of the graph for each value of E_n. Is the normalization of the state consistent with your interpretation?

3. Suppose that one of the terms in the expression at the top of the page were made negative. How would you represent this change on a graph like the one above?

   How would this change affect the physical interpretation you identified in question 2?

Figure A.4: *Representations of Wave functions* in-class tutorial worksheet administered in Autumn 2015 to PHYS 324 (five pages).

II. The position basis



A different visual representation of the state of the particle from the previous page is shown at right.

A. What probabilities can you readily infer about this state from the graph at right? Explain in detail how to determine these values using only the graph.

B. Consider the student discussion below.

Student 1: "The probability of finding the particle is largest where the wave function is positive, and lowest where the wave function is negative."

Student 2: "I disagree. The particle wants to be where the wave function is lowest, so we would be more likely to find it where $\psi(x)$ is very negative."

With which student do you agree, if either? Identify and explain the flaws in the reasoning of each student with whom you disagree.

C. The graph above is a representation of a quantum state in the *position basis,* also known as the *(position-space) wave function*. Explain why these names are appropriate.

D. What are the units on the vertical axis of the graph above? Make sure your answer is consistent with the overall normalization of the state. Explain.

E. The two graphs you have seen in this tutorial represent the same quantum state. How are the two visual representations similar to each other in terms of how they convey information? How are they different? Explain.

✓ Check your results with an instructor.

III. Interference

Consider three particles (A, B, and C) described by the states $|\varphi_A\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, $|\varphi_B\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, and $|\varphi_C\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + i\frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$.

A. *Predict* (without sketching) whether the wave functions in position space associated with each of these three states will be *the same* or *different*. Briefly explain your reasoning.

B. *Predict* (without sketching) whether the *probability densities* in position space associated with each of these three states will be *the same* or *different*. Briefly explain your reasoning.

C. Ask an instructor for a handout showing the wave function and the associated probability density for each particle.

   If your predictions were incorrect, resolve any inconsistencies between the handout and your answers on the previous page. Explain.

D. Suppose the sign (or the complex phase) of a single term in a quantum state written in the energy basis is changed. Indicate whether or not each of the following would be different. Explain.

   1. Energy probabilities

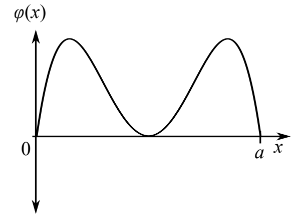
   2. The probability density

The results above indicate that in quantum mechanics, wave functions are subject to interference.

E. Compare the interference between wave functions in quantum mechanics to other examples of interference that you have seen (*e.g.,* for pulses on a spring or for light waves).

IV. Changes of basis

Consider the wave function for a particle in the infinite square well given by $\varphi(x) = Nx(a-x)(2x-a)^2$, where N is the appropriate normalization constant. This wave function is also graphed at right.



A. In what basis is this particle represented? Explain how you determined your answer.

Suppose an energy measurement were made on the particle with the wave function above. Let $P(E_n)$ represent the probability that this measurement results in E_n. Consider the probabilities of the three lowest energy eigenvalues: $P(E_1)$, $P(E_2)$, and $P(E_3)$.

B. *Predict* which of the three probabilities above will be *smallest* and which will be *largest*. Briefly explain (do not write any equations; base your answer on a qualitative argument).

C. Consider the student discussion below.

Student 1: "This wave function is all positive, just like the ground state, so I think that $P(E_1)$ will be the greatest."

Student 2: "I disagree. The wave function looks more like the first excited state, since they both have two bumps, so I think $P(E_2)$ will be the greatest."

Student 3: "I don't know anything about the probabilities for energy because this particle is represented in the position basis. We don't have any information about energy, so the probabilities are all equal."

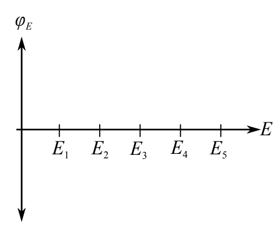
With which student(s), if any, do you agree? For each statement with which you disagree, identify and explain the errors made by that student.

D. Write a mathematical expression for $P(E_n)$, but do not evaluate it.

E. Describe a method for evaluating your expression from question D qualitatively, without calculating any integrals mathematically.

Use your method to check your prediction in question B. Resolve any inconsistencies.

F. Ask an instructor for a handout that will help you to make a graph on the axes at right of this state in the energy basis.



Resolve any inconsistencies with your answers to questions B and E.

G. Consider the student discussion below.

Student 1: "If I know any representation for a quantum state, then I know everything there is to know about it."

Student 2: "I disagree, because if all I know is energy space, then I can't know anything about position space, and vice versa. That's the uncertainty principle."

With which, if either, of the students do you agree? Explain your reasoning.

✓ Check your results with an instructor.

PROBABILITY AMPLITUDE

QM 1

I. Probability amplitude

Suppose that particle A is placed in the infinite square well potential. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$.

A. Write the state of particle A as a sum of the energy eigenstates of the system.

Describe how to determine the coefficient of each term in this sum.

Ask an instructor to check your expression, and to provide a handout that includes the values of the coefficients in the sum above for a particle A. Write a final expression for the state of particle A in the space below.

B. Determine the inner product of the state with itself, $\langle\varphi_A|\varphi_A\rangle$. Show your work.

Does your answer agree with what you expect? Explain.

Figure A.5: *Probability Amplitude* in-class tutorial worksheet administered in Autumn 2015 to PHYS 324 (four pages).

C. Suppose you were to measure the energy of particle A. Which value would be the most likely outcome of this measurement? What is the probability of this outcome? Explain.

D. Explain why it would be *incorrect* to say that $\langle \psi_n | \varphi_A \rangle$ is the probability that particle A is measured to have energy E_n.

The inner product, $\langle \psi_n | \varphi_A \rangle$, between a state that represents a particle, such as $|\varphi_A\rangle$, and an eigenstate associated with an observable, such as $|\psi_n\rangle$, is called a *probability amplitude*.

E. Discuss with your group why the term probability amplitude is appropriate for this inner product.

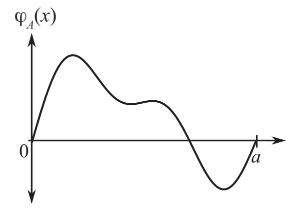
F. Suppose that the value of $\langle \psi_2 | \varphi_A \rangle$ for particle A were changed to $-\frac{1}{\sqrt{2}}$.

1. Is the *probability amplitude* associated with $n = 2$ *the same* or *different?* Explain.

2. Is the *probability* of measuring E_2 *the same* or *different?* Explain.

✓ Discuss your answers with an instructor.

II. Wave functions



The wave function for particle A, $\varphi_A(x)$, is shown at right.

A. Describe how to use wave function to determine the probability that a particle is measured within the region between $x = a/5$ and $x = a/3$.

The wave function for particle A can be written as the following inner product: $\varphi_A(x) = \langle x|\varphi_A\rangle$, where $\langle x|$ is the basis state associated with position x.

B. Would it be appropriate to use the term *probability amplitude* to describe the wave function for particle A? Explain.

C. Use the inner product above to write an expression for the probability that a particle is measured within the region between $x = a/5$ and $x = a/3$.

D. Consider the student discussion below.

Student 1: "When I square the probability amplitude for energy, I get the probability of measuring that energy. Since the wave function is also a probability amplitude, the square of the wave function is also a probability."

Student 2: "I agree that the wave function is a probability amplitude. However, since the eigenvalues of position are continuous, squaring the wave function gives a probability density instead of a probability. That means I need to integrate the square of the probability amplitude to get probability."

With which student do you agree, if either? Explain.

E. Compare the procedure for determining probabilities of energy measurements (section I) to the procedure for determining probabilities of position measurements (section II). How are these procedures similar? How are they different? Explain.

F. Write an expression for the state of particle A, $|\varphi_A\rangle$, in terms of the basis states associated with position, $|x\rangle$. Explain. (*Hint:* What changes do you need to make to the expression in question A of section I?)

Describe how to determine the coefficient of each term in your expression.

Can these coefficients be imaginary? Explain.

✓ Discuss your answers with an instructor.

PROBABILITY AMPLITUDE AND INTERFERENCE

QM 1

I. Probability amplitude in the energy basis

Suppose that particle A is placed in an infinite square well potential of width a. Let the state of the particle at $t = 0$ be given by $|\varphi_A\rangle$ and let the energy eigenstates and eigenvalues of the system be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$.

A. Write the state of particle A as a sum of the energy eigenstates of the system.

Write an expression for the coefficient of each term in this sum using Dirac notation.

B. Describe how the coefficients above are analogous to the projection of a spatial unit vector onto another spatial unit vector (*i.e.*, $\hat{u}_1 \cdot \hat{u}_2$).

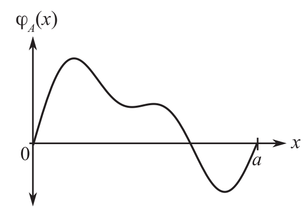
The inner product, $\langle\psi_n|\varphi_A\rangle$, between a state that represents a particle, such as $|\varphi_A\rangle$, and an eigenstate associated with an observable, such as $|\psi_n\rangle$, is called a *probability amplitude*.

C. Describe how the probability amplitude above is related to the probability of an energy measurement outcome.

D. Write an expression *in terms of probability amplitudes* for the probability of measuring the particle to have an energy that is less than E_3, the energy of the second excited state. Explain.

II. Probability amplitude in the position basis

The wave function for particle A, $\varphi_A(x)$, at $t = 0$ is as shown.



A. Describe how to use this wave function to determine the probability that particle A is measured to be in the left half of the well at $t = 0$.

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Figure A.6: *Probability Amplitude* in-class tutorial worksheet administered in Autumn 2016 to PHYS 324 (four pages).

QM 2 ***Probability amplitude and interference***

The *wave function* for particle A can be written as the following inner product: $\varphi_A(x) = \langle x|\varphi_A\rangle$, where $\langle x|$ is the basis state associated with position x.

B. Describe how the probability amplitude above is related to the probability of a position measurement outcome.

Describe how the *probability amplitude in the position basis* is different from the *amplitude* of the wave function.

C. Use the probability amplitude above in Dirac notation to write an expression for the probability that a particle is measured to be in the left half of the well.

Compare the expression you wrote to your expression for the probability of an energy measurement in part D, section I. Describe how they are similar and how they are different.

D. Consider the student discussion below.

Student 1: "When I square the probability amplitude for energy, I get the probability of measuring that energy. Thus, squaring the probability amplitude for position should also give us a probability. It is the probability of finding a particle at a point."

Student 2: "I agree that we should also square the probability amplitude for position. However, since the eigenvalues of position are continuous, we need to integrate the square of the probability amplitude to get the probability of finding a particle in a region."

With which student do you agree, if either? Explain.

E. Compare the procedure for determining probabilities of energy measurements (section I) to the procedure for determining probabilities of position measurements (section II). How are these procedures similar? How are they different? Explain.

In part C, you wrote an expression for the probability that a particle is measured to be in the left half of the well. Now write an analogous expression for the probability that a particle is measured to be moving to the right. Explain.

F. In question A of section I, you wrote an expression for the state of particle A, $|\varphi_A\rangle$, in terms of energy eigenstates. Now write an expression for the same state, $|\varphi_A\rangle$, in terms of the basis states associated with position, $|x\rangle$. Explain.

Describe how to determine the coefficient of each term in your expression.

Give an interpretation for the coefficients above.

G. Do you agree or disagree with the statement below? Explain.

"It seems strange that the coefficient is the wave function, but let me try to make sense of it. If I project the state vector onto the energy basis, each coefficient would be the projection on that energy eigenstate. The square of each coefficient would be the probability of measuring that energy. I can also project the state vector onto the position basis. Then the coefficient, which is the wave function, would be the projection on the position eigenstate. That's why squaring the wave function and doing the integral yields the probability of measuring position."

✓ Discuss your answers with an instructor.

III. Interference

Consider three particles (A, B, and C) described by the states $|\varphi_A\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, $|\varphi_B\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle - \frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$, and $|\varphi_C\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + i\frac{1}{\sqrt{2}}|\psi_2\rangle + \frac{1}{\sqrt{6}}|\psi_4\rangle$ at $t = 0$.

A. *Predict* whether the *probability amplitude in the position basis* associated with each of these three states will be *the same* or *different* at $t = 0$. Briefly explain your reasoning.

B. *Predict* whether the *probability density in the position basis* associated with each of these three states will be *the same* or *different* at $t = 0$. Briefly explain your reasoning.

C. Ask an instructor for a handout showing the probability amplitude and the associated probability density for each particle.

   If your predictions were incorrect, resolve any inconsistencies between the handout and your answers above.

D. Suppose the sign (or the complex phase) of a single probability amplitude for energy in a quantum state is changed. Indicate whether or not each of the following would be different. Explain.

   1. Energy probabilities

   2. The probability amplitude in position space

   3. The probability density in position space

The results above indicate that in quantum mechanics, wave functions are subject to interference.

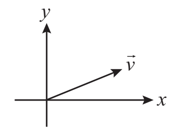
E. Compare the interference between wave functions in quantum mechanics to other examples of interference that you have seen (*e.g.*, for pulses on a spring or for light).

PROBABILITY AMPLITUDE AND INTERFERENCE

QM
1

I. Spatial vectors and projections

Consider a spatial vector $\vec{v}$ shown at right.



A. Write an expression for this vector in terms of the unit vectors $\hat{x}$ and $\hat{y}$.

B. Write an expression for the projection of $\vec{v}$ onto $\hat{x}$ in terms of $\hat{x}$, $\hat{y}$ and $\vec{v}$. (In this tutorial, the projection refers to the scalar component of a vector projection.)

C. Describe how to find the projection of $\vec{v}$ onto an arbitrary vector $\vec{w}$ in the xy-plane.

✓ Discuss your answers with an instructor.

II. Probability amplitudes in the energy basis

Suppose that particle A is placed in an infinite square well potential of width a. Let the state of the particle at $t = 0$ be given by $|\varphi_A\rangle$ and let the energy eigenstates and eigenvalues of the system be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$.

A. Is it possible to write the state of particle A in terms of the energy eigenstates of the system? Explain.

Write $|\varphi_A\rangle$ in terms of the energy eigenstates.

B. The coefficient associated with each energy eigenstate is usually expressed as c_n. Determine c_n using Dirac notation.

Describe how the coefficients above are analogous to the projection of a spatial vector onto a unit spatial vector.

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Figure A.7: *Probability Amplitude* in-class tutorial worksheet administered in Summer 2017 to PHYS 324 (five pages).

QM 2 ***Probability amplitude and interference***

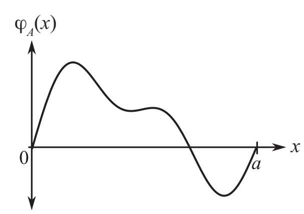
The inner product, $\langle \psi_n | \varphi_A \rangle$, between an the energy eigenstate $|\psi_n\rangle$, and a state that represents a particle, $|\varphi_A\rangle$, is called a *probability amplitude* for state $|\varphi_A\rangle$ to be measured in the energy eigenstate $|\psi_n\rangle$.

C. Describe how to find the probability of an energy measurement outcome using the probability amplitude above.

D. Write an expression *in terms of probability amplitudes in Dirac notation* for the probability of measuring the particle to have an energy that is less than E_4, the energy of the third excited state. Explain.

III. Probability amplitudes in the position basis

The wave function for particle A in section II is $\varphi_A(x)$ at $t = 0$. A graphical representation is as shown at right.



A. Describe how to use this wave function to determine the probability that particle A is measured to be in the left half of the well at $t = 0$.

The *wave function* for particle A can be written as the following inner product: $\varphi_A(x) = \langle x | \varphi_A \rangle$, where $|x\rangle$ is the eigenstate of position operator $\hat{x}$.

B. Give an interpretation of the inner product $\langle x | \varphi_A \rangle$.

Describe how the quantity $\langle x | \varphi_A \rangle$ above is related to the probability of a position measurement outcome.

C. Use the probability amplitudes above in Dirac notation to write an expression for the probability that a particle is measured to be in the left half of the well.

D. Compare the procedure for determining probabilities of energy measurements (section II, part D) to the procedure for determining probabilities of position measurements (section III). How are these procedures similar? How are they different? Explain.

E. Consider the student discussion below.

Student 1: "When I square the probability amplitude $\langle \psi_n | \varphi_A \rangle$ for energy, I get the probability of measuring that energy. Thus, squaring the probability amplitude $\langle x | \varphi_A \rangle$ for position should also give us a probability. It is the probability of finding a particle at a point."

Student 2: "I agree that we should also square the probability amplitude for position. However, since the eigenvalues of position are continuous, we need to integrate the square of the probability amplitude to get the probability of finding a particle in a region."

With which student do you agree, if either? Explain.

F. In question A of section II, you are asked whether or not it is possible to express $|\varphi_A\rangle$ in terms of energy eigenstates. Now is it possible to write an expression for the same state, $|\varphi_A\rangle$, in terms of position eigenstates $|x\rangle$? Explain.

✓ Discuss your answers with an instructor.

G. Now write an expression for the same state, $|\varphi_A\rangle$, in terms of the position eigenstates $|x\rangle$.

(*Hint:* How is this expression analogous to the expression in terms of energy basis?)

Describe how to determine the coefficient of each term in your expression.

Give an interpretation for the coefficients above.

H. Do you agree or disagree with the statement below? Explain.

"It seems strange that the coefficient for each position eigenstate is the wave function, but let me try to make sense of it. If I project the state vector onto the energy basis, each coefficient $\langle\psi_n|\varphi_A\rangle$ would be the projection on that energy eigenstate. The square of each coefficient would be the probability of measuring that energy. I can also project the state vector onto the position basis. Then the coefficient $\langle x|\varphi_A\rangle$, which is the wave function, would be the projection on the position eigenstate. That's why squaring the wave function and doing the integral yields the probability of measuring position."

I. Describe a procedure for determining the probability of measuring an arbitrary observable if the observable has (1) discrete eigenvalues (2) continuous eigenvalues.

✓ Discuss your answers with an instructor.

IV. Supplement: Interference

Consider three particles (A, B, and C) at $t = 0$ described by the states below:

$$\left|\varphi_A\right\rangle = \tfrac{1}{\sqrt{3}}\left|\psi_1\right\rangle + \tfrac{1}{\sqrt{2}}\left|\psi_2\right\rangle + \tfrac{1}{\sqrt{6}}\left|\psi_4\right\rangle$$

$$\left|\varphi_B\right\rangle = \tfrac{1}{\sqrt{3}}\left|\psi_1\right\rangle - \tfrac{1}{\sqrt{2}}\left|\psi_2\right\rangle + \tfrac{1}{\sqrt{6}}\left|\psi_4\right\rangle$$

$$\left|\varphi_C\right\rangle = \tfrac{1}{\sqrt{3}}\left|\psi_1\right\rangle + i\tfrac{1}{\sqrt{2}}\left|\psi_2\right\rangle + \tfrac{1}{\sqrt{6}}\left|\psi_4\right\rangle$$

A. *Predict* whether the quantities below associated with each of these three states will be *the same* or *different* at $t = 0$. Briefly explain your reasoning.

   1. Probability amplitude in the position basis

   2. Probability density in the position basis

B. Ask an instructor for a handout showing the probability amplitude and the associated probability density for each particle.

   If your predictions were incorrect, resolve any inconsistencies between the handout and your answers above.

C. Suppose particle A is instead described by $\left|\varphi'_A\right\rangle = i\tfrac{1}{\sqrt{3}}\left|\psi_1\right\rangle + \tfrac{1}{\sqrt{2}}\left|\psi_2\right\rangle + \tfrac{1}{\sqrt{6}}\left|\psi_4\right\rangle$. Indicate whether or not each of the following would be different from those for $\left|\varphi_A\right\rangle$. Explain.

   1. Energy probabilities

   2. The probability amplitude in position space

   3. The probability density in position space

D. Would your answers above change if the state of particles A is instead described by $\left|\varphi''_A\right\rangle = i\tfrac{1}{\sqrt{3}}\left|\psi_1\right\rangle + i\tfrac{1}{\sqrt{2}}\left|\psi_2\right\rangle + i\tfrac{1}{\sqrt{6}}\left|\psi_4\right\rangle$? Explain.

The results above indicate that in quantum mechanics, wave functions are subject to interference.

E. Does the result of the interference depend on the overall phase? Does it depend on the relative phases? Explain.

REPRESENTATIONS OF WAVE FUNCTIONS

QM 5

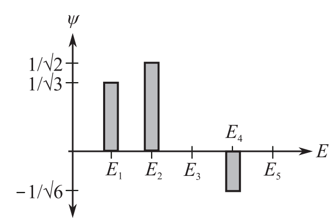
I. The energy basis

Consider a particle in the quantum mechanical infinite square well of width a described by $|\psi\rangle = \frac{1}{\sqrt{3}}|\psi_1\rangle + \frac{1}{\sqrt{2}}|\psi_2\rangle - \frac{1}{\sqrt{6}}|\psi_4\rangle$, where $|\psi_n\rangle$ is the n^{th} energy eigenstate with eigenvalue E_n.

A. List the possible results of an energy measurement on the particle described by the state above and the probability associated with each result. Explain how you determined your answers from the given representation of this quantum state.

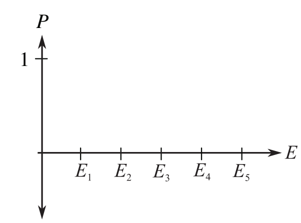
B. The quantum state above is said to be represented in the *energy basis*. Explain why this name is appropriate for this representation.

C. The graph shown at right is a visual representation of the state given above.



1. What quantities do the vertical components of the graph represent?

2. On the graph at right, sketch a histogram for the *probabilities* of energy measurements.



   Describe how you determined the probabilities.

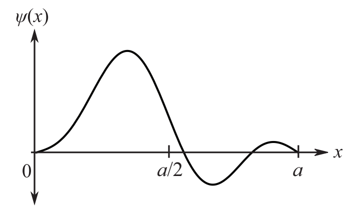
3. Describe how to find the probability that the energy is measured to be smaller than E_3.

   Express your answer using sum notation.

Figure A.8: *Representations of Wave Functions* in-class tutorial worksheet administered in Autumn 2017 to PHYS 324 (four pages).

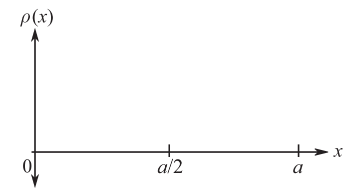
II. Wave function in position space

A different graphical representation of the *same* state of the particle from the previous page is shown at right.



A. What probabilities can you readily infer about this state from the graph at right? Explain.

B. On the graph at right, sketch the probability density for this particle.



C. On the probability density graph, identify and label a geometric quantity that can be used to represent the probability of finding the particle in each of the following regions:

$a/8 \leq x \leq a/4$

$a/4 \leq x \leq 3a/8$

$3a/8 \leq x \leq a/2$

D. Describe how to determine the probability of finding the particle in the region $a/8 \leq x \leq a/2$ using the geometric quantities you identified above.

Express your answer using sum notation.

E. How does your method of finding the probability for position above compare to the method you used for the probability that energy is measured to be *smaller* than E_3 in part C, question 3 on the previous page? Explain.

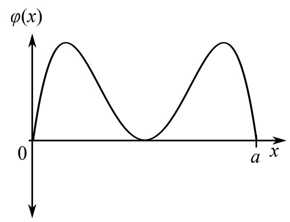
F. For the purpose of determining probabilities, how is the wave function analogous to the coefficients of the state vector written in the energy basis?

How are they different?

✓ Check your results with an instructor.

III. Wave function and energy measurement

Consider the wave function for a particle in the infinite square well given by $\varphi(x) = Nx(a-x)(2x-a)^2$, where N is the appropriate normalization constant. This wave function is also graphed at right.



A. Describe the probabilities you can readily infer from this representation.

Suppose an energy measurement were made on the particle with the wave function above. Let $P(E_n)$ represent the probability that this measurement results in E_n. Consider the probabilities of the three lowest energy eigenvalues: $P(E_1)$, $P(E_2)$, and $P(E_3)$.

B. *Predict* which of the three probabilities above will be *smallest* and which will be *largest*. Briefly explain (do not write any equations; base your answer on a qualitative argument).

C. Consider the student discussion below.

Student 1: "This wave function is all positive, just like the ground state, so I think that $P(E_1)$ will be the greatest."

Student 2: "I disagree. The wave function looks more like the first excited state, since they both have two bumps, so I think $P(E_2)$ will be the greatest."

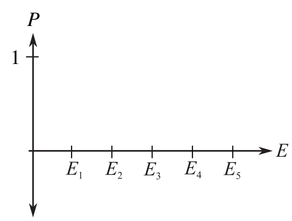
Student 3: "I don't know anything about the probabilities for energy because this state is represented by a wave function in position space. We don't have any information about energy, so the probabilities are all equal."

With which student(s), if any, do you agree? For each statement with which you disagree, identify and explain the errors made by that student.

D. Use the wave function $\varphi(x)$ to write a mathematical expression for $P(E_n)$, but do not evaluate it.

E. Describe a method for evaluating your expression from question D qualitatively, without calculating any integrals mathematically. (*Hint:* What geometric quantity can be used to represent the result of a definite integral?)

F. Ask an instructor for a handout that will help you to make a histogram at right for the probabilities of energy measurements.



Examine the student discussion in question C again and identify any errors made by the students.

G. Consider the student discussion below.

Student 1: "If I know any representation for a quantum state, then I know everything there is to know about it."

Student 2: "I disagree, because if all I know is energy space, then I can't know anything about position space, and vice versa. That's the uncertainty principle."

With which, if either, of the students do you agree? Explain your reasoning.

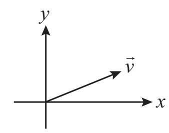
✓ Check your results with an instructor.

PROBABILITY AMPLITUDE

QM 27

I. Spatial vectors and projections

Consider a spatial vector $\vec{v}$ shown at right.



A. Write an expression for this vector in terms of the unit vectors $\hat{x}$ and $\hat{y}$.

B. Write an expression for the projection of $\vec{v}$ onto $\hat{x}$ in terms of the unit vectors and $\vec{v}$. (In this tutorial, the projection refers to the *scalar* component of a vector.)

C. Describe how to find the projection of $\vec{v}$ onto an arbitrary vector $\vec{w}$ in the *xy*-plane.

✓ Discuss your answers with an instructor.

II. Probability amplitudes in the energy basis

Suppose that particle A is placed in an infinite square well potential of width a. Let the state of the particle at $t = 0$ be given by $|\varphi_A\rangle$ and let the energy eigenstates and eigenvalues of the system be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$.

A. Is it possible to write the state of particle A in terms of the energy eigenstates of the system? Explain.

Write $|\varphi_A\rangle$ in terms of the energy eigenstates.

B. The coefficient associated with each energy eigenstate is usually expressed as c_n. Show that c_n can be determined using Dirac notation.

Describe how the coefficients above are analogous to the projection of a spatial vector onto a unit spatial vector.

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 Preliminary First Edition, 2017

Figure A.9: *Probability Amplitude* in-class tutorial worksheet administered in Autumn 2017 to PHYS 324 (four pages).

QM 28 ***Probability amplitude***

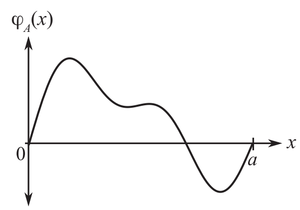
The inner product, $\langle \psi_n | \varphi_A \rangle$, between an the energy eigenstate $|\psi_n\rangle$, and a state that represents a particle, $|\varphi_A\rangle$, is called a *probability amplitude* for state $|\varphi_A\rangle$ to be measured in the energy eigenstate $|\psi_n\rangle$.

C. Describe how to find the probability of an energy measurement outcome using the probability amplitude above.

D. Write an expression *in terms of the probability amplitudes in Dirac notation* for the probability of measuring the particle to have an energy that is less than E_4, the energy of the third excited state. Explain.

III. Probability amplitudes in the position basis

The wave function for particle A in section II is $\varphi_A(x)$ at $t = 0$. A graphical representation is as shown at right.



A. Describe how to use this wave function to determine the probability that particle A is measured to be in the left half of the well at $t = 0$. Express your answer with a mathematical expression and with a graph.

The *wave function* for particle A can be written as the following inner product: $\varphi_A(x) = \langle x | \varphi_A \rangle$, where $|x\rangle$ is the eigenstate of position operator $\hat{x}$.

B. Give an interpretation of the inner product $\langle x | \varphi_A \rangle$.

Describe how wave function $\varphi_A(x)$ is analogous to c_n, the coefficients of state $|\varphi_A\rangle$ written in the energy basis.

G. Write an expression for the same state, $|\varphi_A\rangle$, in terms of the position eigenstates $|x\rangle$. (*Hint:* How is this expression analogous to the expression in terms of energy basis?)

Describe how to determine the coefficient of each term in your expression.

Give an interpretation for the coefficients above.

H. How is the state vector $|\varphi_A\rangle$ different from its corresponding wave function $\varphi_A(x)$?

I. Do you agree or disagree with the statement below? Explain.

"It seems strange that the coefficient for each position eigenstate is the wave function, but let me try to make sense of it. If I project the state vector onto the energy basis, each coefficient $\langle\psi_n|\varphi_A\rangle$ would be the projection on that energy eigenstate. I can also project the state vector onto the position basis. Then the coefficient $\langle x|\varphi_A\rangle$, which is the wave function, would be the projection on the position eigenstate. One can always take inner products to find the coefficients in a particular basis."

I. Describe a procedure for determining the probability of measuring an arbitrary observable if the observable has (1) discrete eigenvalues (2) continuous eigenvalues.

✓ Discuss your answers with an instructor.

C. Use the probability amplitudes above in Dirac notation to write an expression for the probability that a particle is measured to be in the left half of the well.

D. Compare the procedure for determining probabilities above to the procedure for determining the probabilities of energy measurements (section II, part D). How are these procedures similar? How are they different? Explain.

E. Consider the student discussion below.

Student 1: "When I square the probability amplitude $\langle \psi_n | \varphi_A \rangle$ for energy, I get the probability of measuring that energy. Thus, squaring the probability amplitude $\langle x | \varphi_A \rangle$ for position should also give us a probability. It is the probability of finding a particle at a point."

Student 2: "I agree that we should also square the probability amplitude for position. However, since the eigenvalues of position are continuous, we need to integrate the square of the probability amplitude to get the probability of finding a particle in a region."

With which student do you agree, if either? Explain.

F. In question A of section II, you are asked whether or not it is possible to express $|\varphi_A\rangle$ in terms of energy eigenstates. Is it possible to write an expression for the same state, $|\varphi_A\rangle$, in terms of position eigenstates $|x\rangle$? Explain.

✓ Discuss your answers with an instructor.

QUANTUM INTERFERENCE WITH SPIN STATES

QM 1

In this tutorial, you will be considering three ensembles, *A, B,* and *C,* of identical spin-1/2 particles.

I. Superposition states

Consider ensemble *A*. All the particles are prepared in the same way. Let $|\psi_A\rangle$ represent the spin state of each particle. Suppose one thousand of the particles are sent through a Stern-Gerlach apparatus, SG_x, with a non-uniform magnetic field oriented along the *x*-axis. Half of the particles are measured to have spin up in the *x* direction and half of the particles are measured to have spin down in the *x* direction.

A. Write a possible expression for the state $|\psi_A\rangle$ in terms of $|+\rangle_x$ and $|-\rangle_x$. Explain.

B. Is there more than one state that can possibly describe these particles? Explain.

Now consider ensemble *B* with each of the particles prepared in the spin state $|\psi_B\rangle$. Suppose one thousand of the particles are sent through SG_x. The results are the same as for ensemble *A*.

C. Is there an experiment that can be used to distinguish particles in ensemble *A* from particles in ensemble *B?* Explain.

Figure A.10: *Quantum Interference with Spin states* in-class tutorial worksheet administered in Winter 2017 to PHYS 225 (five pages).

QM 2 ***Quantum interference with spin states***

Now consider a different Stern-Gerlach apparatus, SG_z, with a non-uniform magnetic field oriented along the *z*-axis (instead of the *x*-axis as before).

Suppose the particles in ensembles *A* and *B* are sent through SG_z, separately. All of the particles in ensemble *A* are measured to have spin up, while all of the particles in ensemble *B* are measured to have spin down.

D. Write possible expressions for the states $|\psi_A\rangle$ and $|\psi_B\rangle$ in terms of $|+\rangle_x$ and $|-\rangle_x$.

What is the inner product between $|\psi_A\rangle$ and $|\psi_B\rangle$? Explain.

Resolve any inconsistencies with your answers to question B and C.

E. Consider the state $|\phi\rangle = \frac{i}{\sqrt{2}}|+\rangle_x - \frac{i}{\sqrt{2}}|-\rangle_x$. Is this a possible state for the particles in ensemble *B?* Explain.

Rewrite the expression above in terms of $|\psi_B\rangle$. Express any phase factor in the form $e^{i\theta}$.

Is there an experiment that can be used to differentiate between states $|\phi\rangle$ and $|\psi_B\rangle$? (In other words, is there an experiment that can be used to determine the overall phase of a quantum state?) Explain.

F. Consider the student statement below.

"To calculate the probabilities, we always take the absolute squares of the coefficients. Thus, the signs of the coefficients don't matter. All we care about is the magnitudes of the coefficients."

Do you agree with this statement? Explain your reasoning.

✓ Check your results with an instructor.

II. Phase and interference

Consider ensemble *C* with each of the particles prepared in the state $|\psi_C\rangle$. When these particles are sent through SG_x, the results are the same as for particles in ensembles *A* and *B*. However, when they are sent through SG_z, *three quarters* of particles are measured to have spin up in the *z* direction and *one quarter* of the particles are measured to have spin down in the *z* direction.

A. Do you expect the coefficients of the state $|\psi_C\rangle$ in the basis of $|+\rangle_x$ and $|-\rangle_x$ to be the same as the coefficients for $|\psi_A\rangle$ (or $|\psi_B\rangle$)? Explain.

B. Write each coefficient for $|\psi_C\rangle$ in the basis of $|+\rangle_x$ and $|-\rangle_x$ as a positive real number multiplied by a complex exponential (*i.e.*, $r_1 e^{i\alpha}$ and $r_2 e^{i\beta}$). Do you expect the phases α and β to be zero? Explain.

C. What are the magnitudes of these coefficients? (*Hint:* What are the probabilities that the particles in ensemble *C* are measured to have spin up in the *x* direction?)

D. Write the following expressions in terms of the phases α and β.

$_z\langle+|\psi_C\rangle$

$\left|_z\langle+|\psi_C\rangle\right|^2$ (You may find the trigonometric identity, $e^{i(\alpha-\beta)}+e^{-i(\alpha-\beta)}=2\cos(\alpha-\beta)$, helpful.)

E. According to the experimental results described at the start of section II, what is the value of the expression $\left|_z\langle+|\psi_C\rangle\right|^2$?

Use this value to relate the phases α and β.

Simplify the expression for state vector $|\psi_C\rangle$ by factoring out $e^{i\beta}$.

Is it reasonable to let the phase β be zero? Explain.

In this tutorial, you have expressed various spin states as superposition states in the basis of $\left|+\right\rangle_x$ and $\left|-\right\rangle_x$. When the two basis states are superimposed, we say that the basis states interfere with each other, and the phenomenon is called quantum interference.

F. Consider the student dialogue about quantum interference below.

Student 1: "Since probability is a magnitude squared, any phase would not affect the probability. Thus, the phase doesn't change the way that two states interfere with each other."

Student 2: "I disagree. We can always determine the phases by doing experiments. The relative phase between the two basis states $\left|+\right\rangle_x$ and $\left|-\right\rangle_x$ in $\left|\psi_C\right\rangle$ is different than the relative phase in $\left|\psi_A\right\rangle$. That's why those particles have different experimental results when they are sent through SG_z. Interference effects result from phases."

With which student(s), if any, do you agree? Explain your reasoning.

G. In general, quantum state vectors have complex coefficients. Does the result of the interference depend on the overall phase? If so, describe how. If not, explain why not.

Does the result of the interference depend on the relative phase? If so, describe how. If not, explain why not.

✓ Check your results with an instructor.

SUPERPOSITION IN QUANTUM MECHANICS

QM 1

I. Mixed states and superpositions

A. Consider the following statement about a quantum mechanical infinite square well where $\varphi_1(x)$ and $\varphi_2(x)$ are the ground state (n = 1) and first excited state (n = 2) wave functions.

"The wave function given by $\psi(x) = \frac{1}{\sqrt{2}}\left(\varphi_1(x) + \varphi_2(x)\right)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."

Do you agree or disagree with this statement?

B. Consider an ensemble of particles in identical quantum mechanical infinite square wells. Half the particles in this ensemble are in the ground state; the other half are in the first excited state. We will call this the *mixed state ensemble.*

1. Sketch the possible wave functions for a particle in this ensemble.

Suppose you were to select a large number of particles from this ensemble and measure the position of each particle.

2. Will the number of particles you find in the left half of the well be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

Now suppose you were to select *a single particle* at random from this ensemble and measure its position.

3. Is the probability that this particle is found in the left half of the well *greater than, less than,* or *equal to* the probability that it is found in the right half of the well? Explain.

Figure A.11: *Superposition in Quantum Mechanics* in-class tutorial worksheet administered in Winter 2015 and 2016 to PHYS 225 (four pages).

C. Consider a different ensemble of particles in identical quantum mechanical infinite square wells. The wave function for each of the particles in this ensemble is given by $\psi(x)=\frac{1}{\sqrt{2}}\left(\varphi_1(x)+\varphi_2(x)\right)$. We will call this the *superposition ensemble.*

1. Sketch the wave function for a particle in this ensemble.

Suppose you were to select a large number of particles from this ensemble and measure the position of each particle.

2. Will the number of particles you find in the left half of the well be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

Now suppose you were to select *a single particle* at random from this ensemble and measure its position.

3. Is the probability that this particle is found in the left half of the well *greater than, less than,* or *equal to* the probability that it is found in the right half of the well? Explain.

D. If you were to select a single particle from one of the ensembles above at random, would you be able to determine whether the particle came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

E. If instead you were to select a hundred particles from one of the ensembles above at random, would you be able to determine whether the particles came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

F. Consider the statement at the start of section I. Do you agree or disagree with this student? Make sure your answer is consistent with your results in this section.

II. Linear superpositions of eigenstates

Let the energy eigenfunctions and eigenvalues for a system be $\varphi_n(x)$ and E_n, respectively.

These functions satisfy the eigenvalue equation $\hat{H}\,\varphi_n(x) = E_n\,\varphi_n(x)$ where $\hat{H}$ is the Hamiltonian.

A. Consider the wave function given by $\psi(x) = \frac{1}{\sqrt{2}}\big(\varphi_1(x) + \varphi_2(x)\big)$.

1. Is this state an eigenstate of the infinite square well? Explain. (*Hint:* Does it satisfy the eigenvalue equation given above?)

2. Could this state describe a particle in the infinite square well? Explain.

3. Consider the following student discussion.

   Student 1: "This state is a sum of eigenstates which makes it an eigenstate."

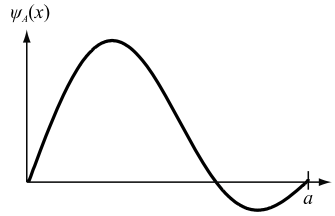
   Student 2: "No, it does not satisfy the eigenvalue equation, so it is not an eigenstate. That means it is not a valid state for a particle in this system."

   Student 3: "You're right that it is not an eigenstate, but it could still represent a particle. Any state that can be written as a sum of eigenstates is valid."

   With which, if any, of the students do you agree? Explain your reasoning. For each student who is incorrect, identify the flaws in that student's reasoning.

4. Write a general expression for an allowed wave function $\psi(x)$ in terms of the energy eigenfunctions of the system, $\varphi_n(x)$. Explain.

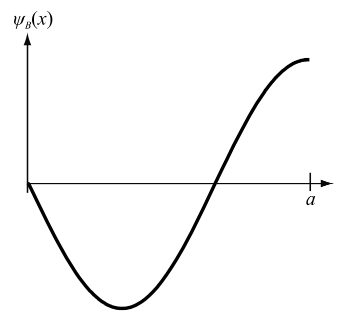
B. Consider the graph shown at right.



1. Could this graph be a possible wave function for a particle in the infinite square well? Explain.

2. Could this graph represent an *energy eigenstate* of the infinite square well? If so, determine which eigenstate. If not, could it be a sum of two or more eigenstates? Explain.

C. Consider a different graph shown at right.



1. Could this graph be a possible wave function for a particle in the infinite square well? Explain.

2. Could this graph represent an *energy eigenstate* of the infinite square well? If so, determine which eigenstate. If not, could it be a sum of two or more eigenstates? Explain.

3. Consider the following discussion between two students.

   Student 1: "$\psi_B(x)$ is an allowed wave function because it is made up of just a single sine function, which means it satisfies the Schrödinger Equation."

   Student 2: "But the boundary conditions for the square well demand that the wave function be zero at the boundaries, and $\psi_B(x)$ does not satisfy that. $\psi_A(x)$ satisfies the boundary condition, so it is a valid state."

   With which student, if either, do you agree? Explain your reasoning.

✓ Check your results with an instructor.

SUPERPOSITION IN QUANTUM MECHANICS

QM 1

In this tutorial, all the wave functions are given at $t = 0$. All the measurements are made at $t = 0$.

I. Mixed states and superpositions

A. Consider the following statement about a quantum mechanical infinite square well where $\psi_1(x)$ and $\psi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) wave functions.

"The wave function given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."

Do you agree or disagree with this statement? Explain.

B. Consider an ensemble of particles in identical quantum mechanical infinite square wells. There is one particle in each square well. Half the particles in this ensemble are in the ground state; the other half are in the first excited state. We call this a *mixed state ensemble*.

1. Sketch the possible wave functions the particles in this ensemble. Explain.

Now suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than*, *less than*, or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left halves of the wells be *greater than*, *less than*, or *equal to* the number of particles you find in the right half of the well? Explain.

Figure A.12: *Superposition in Quantum Mechanics* in-class tutorial worksheet administered in Winter 2017 to PHYS 225 (four pages).

C. Consider a different ensemble of particles in identical quantum mechanical infinite square wells. The wave function for each of the particles in this ensemble is given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$. We call this a *superposition ensemble*.

1. Sketch the possible wave functions for the particles in this ensemble.

Suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than*, *less than*, or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left half of the well be *greater than*, *less than*, or *equal to* the number of particles you find in the right half of the well? Explain.

D. Suppose you were to select a single particle at random from one of the two ensembles above. Would you be able to determine whether the particle came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

E. Suppose instead you were to select a large number of particles at random from one of the two ensembles above. Would you be able to determine whether the particles came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

F. Consider the statement at the start of section I. Do you agree or disagree with this student? Make sure your answer is consistent with your results in this section.

II. Superposition and Interference

Consider two superposition ensembles, *A* and *B*, of identical particles in identical infinite square wells. There is one particle in each square well. The particles in ensemble *A* are all described by the wave function $\psi_A(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_3(x)\right)$. The particles in ensemble *B* are all described by the wave function $\psi_B(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + i\psi_3(x)\right)$.

A. Is there an experiment that can be used to distinguish the particles in ensemble *A* from particles in ensemble *B?* Explain.

B. Consider the student statement below.

"The probability of measuring either E_1 or E_3 is 1/2 for both of the ensembles. Thus, there is no way to distinguish them. The imaginary number goes away when we take the absolute square."

Do you agree with this statement? Explain your reasoning.

C. Sketch the wave function in position space for a particle in ensemble *A*.

D. Sketch the probability density in position space for the particle in ensemble *A*.

E. Sketch the wave function in position space for a particle in ensemble *B*. Represent the real part with a solid line and the imaginary part with a dashed line.

F. Sketch the probability density in position space for the particle in ensemble *B*.

Is the probability density for a particle in ensemble *B the same* or *different than* the probability density for a particle in ensemble *A?*

G. Suppose you were to select a single particle at random from one of the two ensembles above and measure its position. Would you be able to determine whether the particle came from ensemble *A* or ensemble *B?* Explain.

H. Suppose instead you were to select a large number of particles at random from one of the two ensembles above and measure their positions. Would you be able to determine whether the particles came from ensemble *A* or ensemble *B?* Explain.

Resolve any inconsistencies with your answers to questions A and B.

SUPERPOSITION IN QUANTUM MECHANICS

In this tutorial, assume that all the wave functions are given at $t = 0$. All the measurements are made at $t = 0$.

I. Mixed states and superposition states

A. Consider the following statement about a quantum mechanical infinite square well, where $\psi_1(x)$ and $\psi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) wave functions.

"The wave function given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."

Do you agree or disagree with this statement? Explain.

The first two pages of this tutorial are intended to guide you to understand the difference between mixed and superposition states. Then, we revisit the statement above and reflect on the proper interpretation of the state.

B. Consider an ensemble of identical particles each in an identical, quantum mechanical, infinite square well. Half the particles in this ensemble are in the ground state; the other half are in the first excited state. We call this a *mixed state ensemble*.

1. Sketch the possible wave function(s) for the particles in this ensemble. Explain.

Suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than, less than,* or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left halves of the wells be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

Figure A.13: *Superposition in Quantum Mechanics* in-class tutorial worksheet administered in Summer 2017 to PHYS 225 (four pages)

C. Consider a different ensemble of identical particles each in identical infinite square well. The wave function for each of the particles in this ensemble is given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$. We call this a *superposition ensemble*.

1. Sketch the possible wave function(s) for the particles in this ensemble.

Suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than, less than,* or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left half of the well be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

D. Below, we compare the results of making random measurements on the two different ensembles.

1. Suppose you were to select a single particle at random from one of the two ensembles above. Would you be able to determine whether the particle came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

2. Suppose instead you were to select a large number of particles at random from one of the two ensembles above. Would you be able to determine whether the particles came from the *mixed state ensemble* or the *superposition ensemble?* Explain.

E. Consider the statement at the start of section I. Do you agree or disagree with this statement? Explain how your answer is consistent with your results in this section.

II. Superposition and Interference

Consider two superposition ensembles, *A* and *B*, of identical particles each in an identical infinite square well. The wave functions of particles in ensembles *A* and *B* are given below, where $\psi_1(x)$ and $\psi_3(x)$ are the ground state ($n = 1$) and second excited state ($n = 3$) energy eigenfunctions.

$$\psi_A(x) = \tfrac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_3(x)\right) \qquad \psi_B(x) = \tfrac{1}{\sqrt{2}}\left(\psi_1(x) + i\psi_3(x)\right)$$

A. Is there an experiment that can be used to distinguish the particles in ensemble *A* from the particles in ensemble *B?* Explain.

B. Consider the statement below.

"The probability of measuring either E_1 or E_3 is 1/2 for both of the ensembles. Thus, there is no way to distinguish them. The imaginary number goes away when we take the modulus square."

Do you agree with this statement? Explain your reasoning.

C. Sketch the wave function in position space for a particle in ensemble *A* and a particle in ensemble *B*. If necessary, sketch the real and imaginary parts separately.

Ensemble *A*	Ensemble *B*

D. Sketch the probability density in position space for the particle in ensemble *A* and a particle in ensemble *B*.

Ensemble *A*	Ensemble *B*

E. Is the probability density for a particle in ensemble *B* *same as* or *different from* the probability density for a particle in ensemble *A?*

F. Suppose you were to select a large number of particles from one of the two ensembles above at random and were to measure their positions. Would you be able to determine whether the particles came from ensemble *A* or ensemble *B?* Explain.

Resolve any inconsistencies with your answers to questions A and B.

G. Suppose the particles in ensemble *B* were instead described by the wave function $\psi'_B(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) - \psi_3(x)\right)$. Would you be able to distinguish particles in ensemble *A* from particles in ensemble *B?* Explain.

H. What is the relative phase between the energy eignfunctions $\psi_1(x)$ and $\psi_3(x)$ in each of the wave functions $\psi_A(x)$, $\psi_B(x)$, and $\psi'_B(x)$? (*Hint:* The phase of a complex number can be considered as the angle from the positive real axis in a complex plane.)

I. Based on your answers above, does probability density depend on the relative phase between the energy eigenfunctions $\psi_1(x)$ and $\psi_3(x)$? Explain.

SUPERPOSITION IN QUANTUM MECHANICS

QM 1

In this tutorial, assume that all the wave functions are given at $t = 0$. All the measurements are made at $t = 0$.

I. Mixed ensemble and pure ensemble

A. Consider the following statement about a quantum mechanical infinite square well, where $\psi_1(x)$ and $\psi_2(x)$ are the ground state ($n = 1$) and first excited state ($n = 2$) wave functions.

"The wave function given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$ represents a lack of knowledge about the state of the system. The system is definitely in either the ground state or the first excited state. The wave function simply tells you that the probability is 1/2 that the system is really in the ground state and 1/2 that it is really in the first excited state."

Do you agree or disagree with this statement? Explain.

The first two pages of this tutorial are intended to guide you to understand the difference between mixed and pure ensembles. Then, we revisit the statement above and reflect on the proper interpretation of the state.

B. Consider an ensemble of identical particles each in an identical, quantum mechanical, infinite square well. Half the particles in this ensemble are in the ground state; the other half are in the first excited state. We call this a *mixed ensemble*.

1. Sketch the possible wave function(s) for the particles in this ensemble. Explain.

Suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than, less than*, or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left halves of the wells be *greater than, less than*, or *equal to* the number of particles you find in the right half of the well? Explain.

Figure A.14: *Superposition in Quantum Mechanics* in-class tutorial worksheet administered in Winter 2018 to PHYS 225 (four pages).

C. Consider a different ensemble of identical particles each in identical infinite square well. The wave function for each of the particles in this ensemble is given by $\frac{1}{\sqrt{2}}\left(\psi_1(x)+\psi_2(x)\right)$. We call this a *pure ensemble* since each particle is in the same superposition state.

1. Sketch the possible wave function(s) for the particles in this ensemble.

Suppose you were to select *a single particle* at random from this ensemble and measure its position.

2. Is the probability that this particle would be found in the left half of the well *greater than, less than,* or *equal to* the probability that it would be found in the right half of the well? Explain.

Now suppose you were to select *a large number of particles* at random from this ensemble and measure the position of each particle in each square well.

3. Would the number of particles you find in the left half of the well be *greater than, less than,* or *equal to* the number of particles you find in the right half of the well? Explain.

D. Below, we compare the results of making random measurements on the two different ensembles.

1. Suppose you were to select a single particle at random from one of the two ensembles above. Would you be able to determine whether the particle came from the *mixed ensemble* or the *pure ensemble?* Explain.

2. Suppose instead you were to select a large number of particles at random from one of the two ensembles above. Would you be able to determine whether the particles came from the *mixed ensemble* or the *pure ensemble?* Explain.

E. Consider the statement at the start of section I. Do you agree or disagree with this statement? Explain how your answer is consistent with your results in this section.

II. Superposition and Interference

Consider two pure ensembles, A and B, of identical particles each in an identical infinite square well. The wave functions of particles in ensembles A and B are given below, where $\psi_1(x)$ and $\psi_3(x)$ are the ground state ($n = 1$) and second excited state ($n = 3$) energy eigenfunctions.

$$\psi_A(x) = \tfrac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_3(x)\right) \qquad \psi_B(x) = \tfrac{1}{\sqrt{2}}\left(\psi_1(x) + i\psi_3(x)\right)$$

A. Is there an experiment that can be used to distinguish the particles in ensemble A from the particles in ensemble B? Explain.

B. Consider the statement below.

"The probability of measuring either E_1 or E_3 is 1/2 for both of the ensembles. Thus, there is no way to distinguish them. The imaginary number goes away when we take the modulus square."

Do you agree with this statement? Explain your reasoning.

C. Sketch the wave function in position space for a particle in ensemble A and a particle in ensemble B. If necessary, sketch the real and imaginary parts separately.

Ensemble A	Ensemble B

D. Sketch the probability density in position space for the particle in ensemble A and a particle in ensemble B.

Ensemble A	Ensemble B

E. Is the probability density for a particle in ensemble *B* *same as* or *different from* the probability density for a particle in ensemble *A?*

F. Suppose you were to select a large number of particles from one of the two ensembles above at random and were to measure their positions. Would you be able to determine whether the particles came from ensemble *A* or ensemble *B?* Explain.

Resolve any inconsistencies with your answers to questions A and B.

G. Suppose the particles in ensemble *B* were instead described by the wave function $\psi'_B(x) = \frac{1}{\sqrt{2}}\left(\psi_1(x) - \psi_3(x)\right)$. Would you be able to distinguish particles in ensemble *A* from particles in ensemble *B?* Explain.

H. What is the relative phase between the energy eigenfunctions $\psi_1(x)$ and $\psi_3(x)$ in each of the wave functions $\psi_A(x)$, $\psi_B(x)$, and $\psi'_B(x)$? (*Hint:* The phase of a complex number can be considered as the angle from the positive real axis in a complex plane.)

Based on your answers above, does probability density depend on the relative phase between the energy eigenfunctions $\psi_1(x)$ and $\psi_3(x)$? Explain.

I. In general, is it possible to determine a relative phase based on experimental results? Explain.

TIME DEPENDENCE IN QUANTUM MECHANICS

QM 1

I. Energy eigenstates of the infinite square well

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ \infty & x > a \end{cases}$$

Consider the quantum mechanical infinite square well potential, $V(x)$, defined at right. The energy eigenvalues for this potential are given by $E_n = n^2 E_1$. At time $t = 0$ a particle is in the ground state; assume the wave function is entirely real at $t = 0$.

A. Sketch and label the wave function for this particle at $t = 0$.

B. How does the wave function for this particle change with time? Write the real and imaginary parts of the wave function as functions of time. Explain.

One way to graph one-dimensional, complex-valued functions (such as wave functions) is to plot the real part of the function, the imaginary part of the function, and the spatial dimension on three orthogonal axes. Ask an instructor for tools to help you visualize this representation.

C. Sketch and label the wave function for this particle at $t_1 > 0$. Use the visualization tool provided by the instructor to help illustrate the time dependence.

D. How is the wave function at $t = t_1$ similar to and different from the wave function at $t = 0$? Explain.

Consider a second particle in the first excited state of the infinite square well. Assume its wave function is also entirely real at $t = 0$.

E. Does the wave function for the second particle change *faster*, *slower*, or *at the same rate* as the first particle? Explain. (*Hint:* Consider the energy associated with each particle.)

Figure A.15: *Time Dependence in Quantum Mechanics* in-class tutorial worksheet administered in Autumn 2015 to PHYS 324 (four pages).

F. Consider the first instant, $T > 0$, when the graph of the wave function for the *second particle* (in the first excited state) looks exactly the same as it did at $t = 0$.

1. Determine the value of T in terms of E_1, the energy of the ground state.

2. Sketch the wave function for the *first particle* (the ground state) at this new time, $t = T$.

3. Sketch both wave functions at time $t = 2T$.

4. Use a visualization tool to demonstrate the time dependence of these states.

G. Sketch the probability density for both particles at $t = 0$.

Do the probability densities for these states depend on time? If so, how?

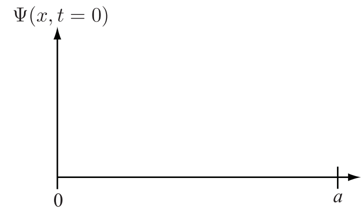
H. Not all probability densities are independent of time. Why do the wave functions you have considered in this section correspond to probability densities that do not depend on time?

✓ Check your results with an instructor.

II. Linear superpositions of energy eigenstates

Consider a particle in a quantum mechanical infinite square well that is identical to the well in section I. The wave function at $t = 0$ is given by $\Psi(x, t=0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$, where $\psi_n(x)$ is the energy eigenfunction associated with the energy $E_n = n^2 E_1$.

A. In the space at right, sketch the wave function for this particle at $t = 0$.



B. Does the wave function for this particle depend on time? Explain.

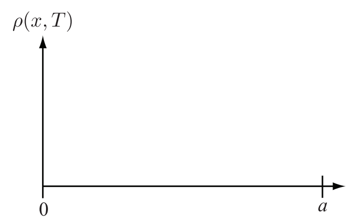
C. In the space at right, sketch the probability density for this particle at $t = 0$.

D. *Predict* whether the probability density for this particle depends on time. Explain.

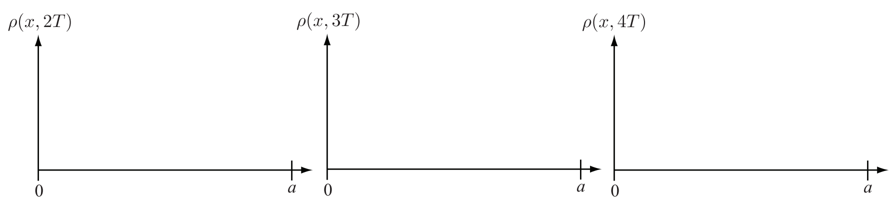
E. Is this particle in an energy eigenstate? Explain.

F. Recall the definition for T from section I. In the space below, sketch the real and imaginary parts of the wave function for this particle at $t = T$. You may find your visualization tool helpful.

G. In the space at right, sketch the probability density at $t = T$. Explain.



H. Sketch the probability density at each of the following times: $2T$, $3T$, and $4T$.



I. Are there times between T and $2T$ when the probability density is the same as it was at $t = 0$? Use your visualization tool to illustrate your answer.

J. Obtain a handout showing the time evolution of the probability density. Resolve any inconsistencies with your answers above.

K. What will happen to the state after a *very long time* has elapsed? Explain.

L. What must be true of a quantum mechanical state if the probability density associated with it is to change with time? Explain.

✓ Check your results with an instructor.

TIME DEPENDENCE IN QUANTUM MECHANICS

I. Energy eigenstates of the infinite square well

$$V(x) = \begin{cases} \infty & x < 0 \\ 0 & 0 \le x \le a \\ \infty & x > a \end{cases}$$

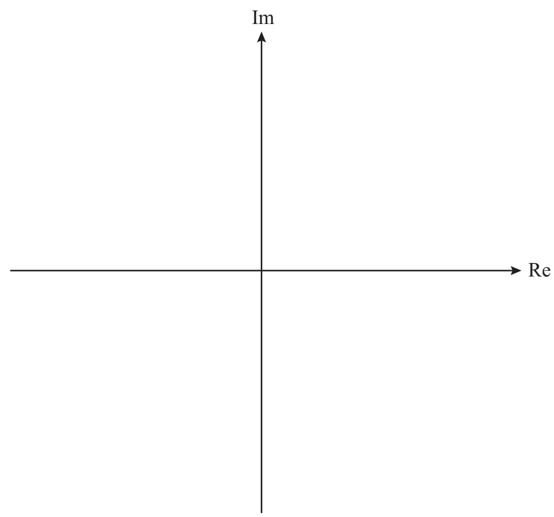
Consider the quantum mechanical infinite square well potential, $V(x)$, defined at right. The energy eigenvalues for this potential are given by $E_n = n^2 E_1$. At time $t = 0$ a particle is in the ground state; assume the wave function is *entirely real* at $t = 0$.

A. Sketch and label the wave function for this particle at $t = 0$.

B. How does the wave function for this particle change with time, if at all? Explain.

One way to visualize one-dimensional, complex-valued wave functions is to place the x-axis of the wave function graph orthogonal to the complex plane such that the x-axis points out of the page. Ask an instructor for tools to help you visualize this representation.

C. Use the set of axes below and the visualization tool provided by the instructor to illustrate the wave function for this particle at $t = 0$.



On the complex plane above, sketch a vector to indicate the *phase* of the wave function at $t = 0$. For example, if the phase of the wave function is π at some instant, the vector should point along the negative real axis.

D. Use the visualization tool provided by the instructor to illustrate the wave function for this particle at $t_1 > 0$, a short time later. Record the phase at this instant by sketching the corresponding vector on the diagram above.

Figure A.16: *Time Dependence in Quantum Mechanics* in-class tutorial worksheet administered in Autumn 2017 to PHYS 324 (five pages).

E. How is the wave function at $t = t_1$ similar to the wave function at $t = 0$? How is it different? Explain.

F. In the space below, sketch the probability density for the particle at $t = t_1$.

G. Is there an instant in time at which the probability density has a different shape than it does at $t = t_1$? Explain.

H. Not all probability densities are independent of time. Explain how is the time-dependent wave function you have considered in this section consistent with the fact that the probability density does not depend on time?

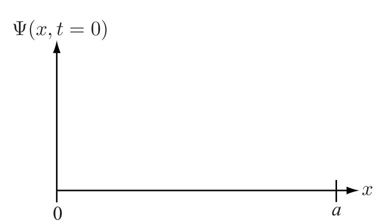
✓ Check your results with an instructor.

II. Linear superpositions of energy eigenstates

Consider a different particle in the quantum mechanical infinite square well described in section I. The wave function at $t = 0$ is given by $\Psi(x,t=0) = \frac{1}{\sqrt{2}}\left(\psi_1(x) + \psi_2(x)\right)$, where $\psi_n(x)$ is the energy eigenfunction associated with the energy $E_n = n^2 E_1$.

A. Is this particle in an energy eigenstate? Explain.

B. In the space at right, sketch the wave function for this particle at $t = 0$.



Does the wave function for this particle depend on time? Explain.

C. In the space at right, sketch the probability density for this particle at $t = 0$.

Predict whether the probability density for this particle depends on time. Explain.

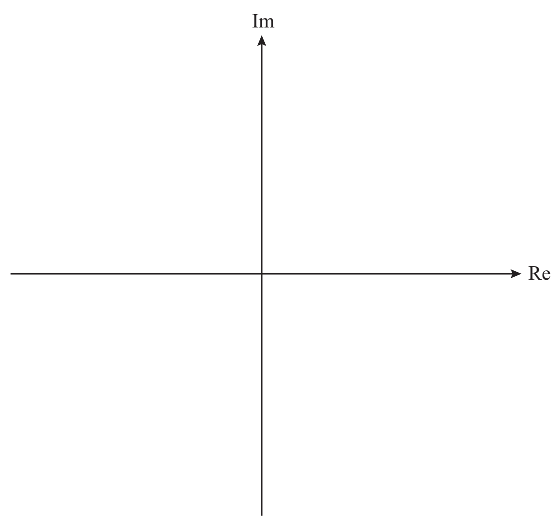
D. Write an expression for the time-dependent wave function $\Psi(x,t)$.

Does $\psi_2(x)$ change *faster, slower,* or *at the same rate* as $\psi_1(x)$? Explain.

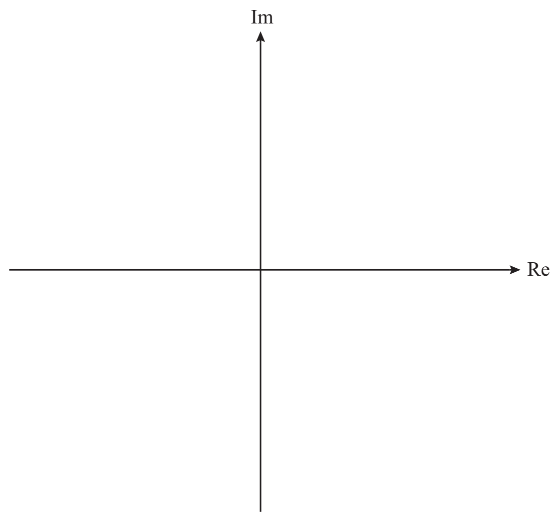
E. Consider the first instant, $T > 0$, when $\psi_2(x)$ looks exactly the same as it did at $t = 0$.

1. Determine the value of T in terms of E_1, the energy of the ground state.

2. Use the visualization tool provided by the instructor to illustrate *each* of the energy eigenfunctions at $t = \mathrm{T}$. Record the phase for each of the energy eigenfunctions.

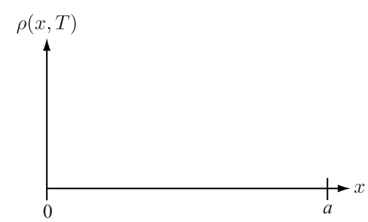


3. Use the visualization tool provided by the instructor to illustrate each of the energy eigenfunctions at $t = 2T$. Record the phase for each of them.

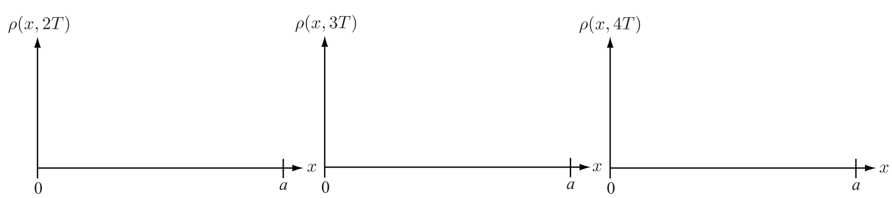


F. In the space below, sketch the real and imaginary parts of $\Psi(x,T)$, the wave function for this particle at $t = T$.

G. In the space at right, sketch the probability density at $t = T$. Explain.



Sketch the probability density at each of the following times: $2T$, $3T$, and $4T$.



H. *Predict* whether there are times between T and $2T$ when the probability density is the same as it is at $t = 0$. Use your visualization tool to illustrate your answer.

I. Obtain a handout showing the time evolution of the probability density. Resolve any inconsistencies with your answers above.

J. Suppose both $\psi_1(x)$ and $\psi_2(x)$ have the same phase, $2\pi/3$, at an unknown instant $t = t_u$. Would the probability density for this particle at $t = t_u$ be the same as it is at $t = 0$? Explain.

What is the relative phase between $\psi_1(x)$ and $\psi_2(x)$ when the probability density is the same as it is at $t = 0$? Explain.

K. Describe how the relative phase affects the time dependence of the probability density.

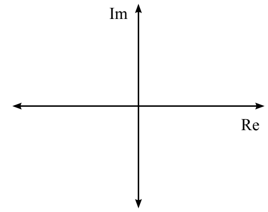
L. What must be true of a quantum mechanical state if the probability density associated with it is to change with time? Explain.

✓ Check your results with an instructor.

TWO-STATE TIME DEPENDENCE

QM 1

I. Complex numbers



Consider the complex number $z = 1 - i$.

A. Plot this number on the set of axes shown at right.

B. Draw a vector from the origin to the plotted number.

1. What does the length of this vector represent about the complex number *z?* Determine the value in this case.

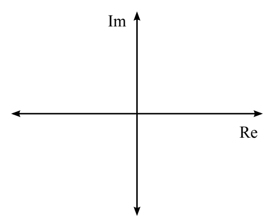
2. What does the angle between the real axis and this vector represent about the complex number *z?* Determine the value in this case. (*Note:* Is the angle positive or negative?)

3. Rewrite *z* as a complex exponential. Describe how you used the numbers you determined in the previous two questions.

Consider the complex function of time $f(t) = e^{-2it}$.

C. Find the smallest positive time, $T > 0$, such that $f(T) = f(0)$ (this time is known as the period). (*Hint:* Recall that $e^{\pm 2\pi i} = 1$)

D. Graph $f(t)$ on the axes at right, following the steps below.



1. Plot $f(t)$ for the points $t = 0$, $T/8$, $T/4$, $T/2$, $3T/4$, and T.

2. Connect these points with a dashed line. Be sure to indicate the direction of increasing *t*.

3. As *t* increases, does the length of the vector between the origin and the point described by $f(t)$ change? If so, describe how it changes.

4. As *t* increases, does the angle between the positive real axis and the vector from the origin to the point described by $f(t)$ change? If so, describe how it changes.

✓ Discuss your results with an instructor.

Figure A.17: *Two-state Time Dependence* in-class tutorial worksheet administered in Winter 2016 to PHYS 225 (four pages).

II. Time dependence for a quantum state

Consider an electron in the spin state $|\chi\rangle \equiv \begin{pmatrix}\alpha \\ \beta\end{pmatrix}$. (*Note:* when no basis is given, use the z-basis.)

A. Determine the inner product ${}_x\langle + | \chi\rangle$ in terms of α and β.

B. Is this inner product related to a probability? If so, which probability? If not, explain why not.

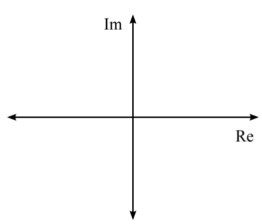
C. An electron's spin is measured just before $t = 0$ and determined to be $+\hbar/2$ in the ***x*-direction**.

1. Determine the state of this electron as a column vector at $t = 0$. Explain.

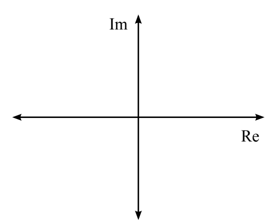
At $t = 0$, a constant magnetic field B_o is turned on in the ***z*-direction**. Recall that, at $t > 0$, the state of a particle in a magnetic field is given by $|\chi(t)\rangle \equiv \begin{pmatrix}\alpha(t) \\ \beta(t)\end{pmatrix} = \begin{pmatrix}\alpha(0)e^{-i\omega_o t/2} \\ \beta(0)e^{+i\omega_o t/2}\end{pmatrix}$, where ω_o depends on the strength of both the magnetic field and the particle's magnetic moment.

2. *For the electron described above*, graph $\alpha(t)$ and $\beta(t)$ on the axes below, following the procedure from the previous section. Indicate the phase and the magnitude of $\alpha(t)$ and $\beta(t)$ at several instants in time (*e.g.*, $t = T/8$, $t = T/4$, *etc*.).

$\alpha(t)$



$\beta(t)$

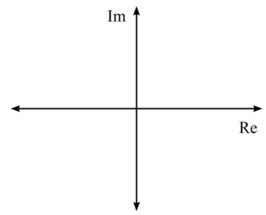


Describe how $\alpha(t)$ and $\beta(t)$ are changing in time. Discuss any similarities and differences between them.

Are $\alpha(t)$ and $\beta(t)$ changing in time at the same rate?

3. Write an expression for the inner product ${}_x\langle + | \chi(t) \rangle$ as a function of time.

4. On the graph at right, plot the inner product ${}_x\langle + | \chi(t) \rangle$ at several different instants (*e.g.*, $t = T/8$, $t = T/4$, *etc.*) for this particle. (*Hint:* Express the inner product in terms of trig. functions)



D. Suppose that you were to measure the ***x*-component** of the electron's spin at a later time, $t > 0$.

1. Does the probability that the result of this measurement is $+\hbar/2$ depend on time? Explain.

2. Use your graph of ${}_x\langle + | \chi \rangle$ to determine the smallest time $t > 0$ at which the probability of measuring $+\hbar/2$ in the x-direction is *equal to* the probability at $t = 0$. Explain.

   How does this time compare to T, the period of $\alpha(t)$ and $\beta(t)$?

3. Write an expression for the probability of measuring $+\hbar/2$ in the x-direction in terms of $\alpha(t)$ and $\beta(t)$.

Suppose instead that you were interested in the probability that the ***z*-component** of spin is equal to $+\hbar/2$. Assume that no measurement was made in the previous question.

4. Write an expression for this probability in terms of $\alpha(t)$ and $\beta(t)$. Does this probability depend on time? Explain. (*Hint:* Rewrite $\alpha(t)$ and $\beta(t)$ in terms of $\alpha(0)$ and $\beta(0)$ for this electron.)

E. Use your answers from this section to fill out the first two columns of the handout.

✓ Discuss your results with an instructor.

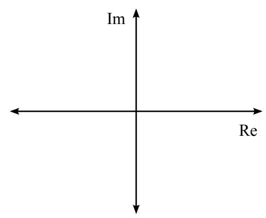
III. Time dependence for a stationary state

Now suppose that an electron's spin is measured and determined to be $+\hbar/2$ in the z-direction. As in the previous section, a constant magnetic field B_o is turned on in the z-direction at $t = 0$

A. Determine the state of this electron as a function of time in the form $\begin{pmatrix} \alpha(t) \\ \beta(t) \end{pmatrix}$. (*Hint:* What is $\alpha(t)$ at $t = 0$?)

B. Graph $\alpha(t)$ on the set of axes at right by following the procedure from the previous sections.

C. What is the probability that a measurement of the z-component of this electron's spin results in $+\hbar/2$? Justify your answer using your graph.



D. What is the probability that a measurement of the ***x*-component** of this electron's spin results in $+\hbar/2$? (Assume no measurements were made in the previous question.)

E. Which of the probabilities in this section, if any, depend on time? Explain. Use your answers to fill out the rest of the handout.

The term *stationary state* may be used to describe the state of the electron in this section.

F. Justify the use of this term for this state. For a stationary state, does the state depend on time?

G. Consider the student discussion below.

Student 1: "The probability of measuring the spin in the z-direction was independent of time for both of the states we considered in this tutorial, so they are both stationary states."

Student 2: "I disagree. The state that started as spin up in the z-direction is stationary because *all* probabilities are independent of time, even though the state does depend on time."

Student 3: "I think it depends on what we measure. Any state is stationary if we measure the spin in the z-direction, but not stationary if we were measure the spin in the x-direction."

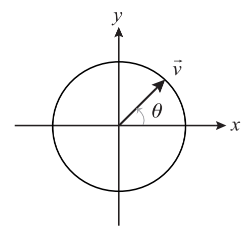
With which student(s) do you agree, if any? For each student who is incorrect, identify the flaw(s) in that student's reasoning. Explain.

✓ Discuss your results with an instructor.

PHASORS PO 1

I. Rotating vectors

Consider a unit vector $\vec{v}$ in the xy-plane as shown at right.
(*Note:* Recall that the magnitude of a unit vector is equal to 1.)

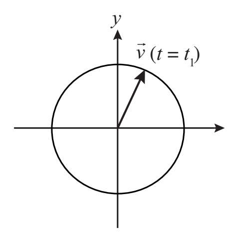


A. Write an expression for the y-component, v_y, of vector $\vec{v}$ in terms of the angle θ shown in the figure.

B. Suppose the unit vector $\vec{v}$ is rotating counter-clockwise in the xy-plane with a constant angular speed ω_o. Let $\vec{v}$ point entirely in the positive x-direction at $t = 0$. Write an expression for v_y, the y-component of the vector as a function of time.

Describe in words how v_y changes as t increases.

C. The graph at right shows $\vec{v}$ at some time $t = t_1$. Now consider another vector, $\vec{u}$, which could be added to $\vec{v}$ so that the sum is equal to zero. Sketch vector $\vec{u}$ on the graph at right.



Would it be correct to say that the y-components of $\vec{u}$ and $\vec{v}$ cancel at $t = t_1$? Explain.

D. Suppose that $\vec{u}$ rotates in the *same* direction as $\vec{v}$ (*i.e.*, counter-clockwise), with the *same* angular speed, ω_o.

1. Do the y-components of $\vec{u}$ and $\vec{v}$ cancel at *all instants in time?* Describe how you determined you answer.

2. By what angle is $\vec{u}$ ahead of $\vec{v}$ at $t = t_1$? Give your answer in both radians and degrees.

3. Does this angle change in time? Explain.

E. Use your expression for v_y from question B and the angle you determined in part D to write an expression for u_y, the y-component of $\vec{u}$.

Figure A.18: *Phasors* in-class tutorial worksheet administered in Summer 2016 to PHYS 123 (five pages).

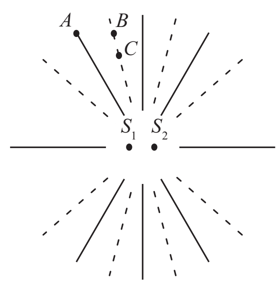
F. What is the *phase difference* $\Delta\varphi$ between the two sine functions u_y and v_y? Explain.

Compare the *phase difference* to the *angle* between the two vectors.

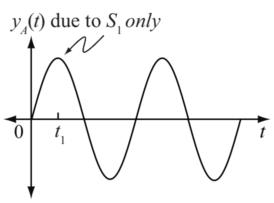
Any sinusoidal function, such as $y(t) = \sin(\omega t)$ can be represented by the y-component of a rotating vector. Such a representation is sometimes called a *phasor*. The phase difference, $\Delta\varphi$, between two sinusoidal functions, such as $\sin(\omega t)$ and $\sin(\omega t + \Delta\varphi)$, can be regarded as the angle between the corresponding phasors.

II. Phasors and water waves

Consider two point sources of water waves, S_1 and S_2. The diagram at right shows the nodal lines (dashed) and lines of maximum constructive interference (solid) due to these two sources.

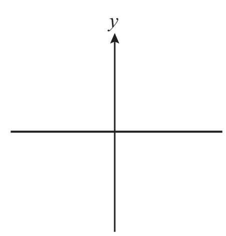


A. The graph below shows the displacement of the water surface versus time at point A due to source S_1 *only*.



1. On the graph above, sketch the displacement over time at point A due to source S_2 only. Explain.

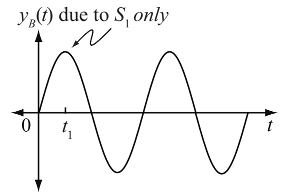
2. On the diagram at right, sketch a pair of phasors to represent the displacement at point A due to each of the two sources at $t = t_1$. (*Note:* t_1 is labeled on the graph above.) How do the phasors at point A change over time? Explain.



3. What is the angle between the phasors at $t = t_1$? Does this angle change over time?

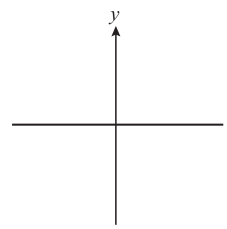
Do you expect this angle to be different at a different point along the line of maximum constructive interference? Explain.

B. The graph at right shows the displacement versus time at point B due to source S_1 *only*.



1. On the graph, sketch the displacement over time at point B due to source S_2 only. Explain.

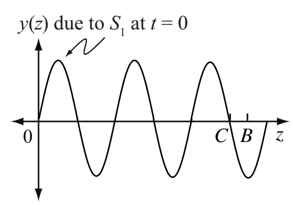
2. Sketch a pair of phasors to represent the displacement at point B due to each of the two sources $t = t_1$. How do the phasors change over time? Explain.



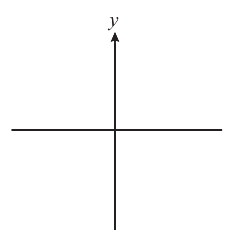
3. What is the angle between the phasors at $t = t_1$? Does it change over time?

   Do you expect this angle to be different at a different point along the nodal line? Explain.

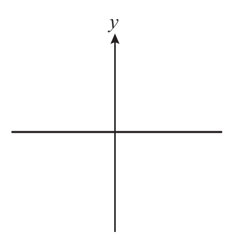
C. Now consider the entire nodal line along which points B and C lie. The graph at right shows the displacement of the water surface along this nodal line due to source S_1 *at a single instant in time* ($t = 0$). In this graph, the horizontal axis represents the distance z from the source S_1.



1. On the graph, sketch the displacement of the water surface along this nodal line at time $t = 0$ due to source S_2 only. Explain.



2. Sketch a pair of phasors to represent the displacement at point B due to each of the two sources at $t = 0$. What is the angle between them? Explain.



3. Sketch a pair of phasors to represent the displacement at point C due to each of the two sources at $t = 0$. What is the angle between them? Explain.

4. Does the angle between the phasors due to each source change along the nodal line? Explain.

5. For each phasor, does the direction in which the phasor is pointing depend on the distance from the source? Explain.

In this section, you used phasors to represent waves as they propagate through a medium, changing in both time and space. Such a wave can also be represented using an equation such as $y = \sin(kz + \omega t)$. The angle between the phasor and the x-axis is then given by the argument of the sine function (*e.g.*, $\theta = kz + \omega t$ in the given example).

C. Do you think it would be appropriate to use phasors to represent light waves? Explain why or why not.

✓ Check your results with an instructor.

III. Phasors and multiple-slit interference

Suppose that coherent red light were incident on a mask with three equally spaced narrow slits.

A. Sketch three phasors that could correspond to the light passing through the slits at a principal maximum on the screen.

What is the angle between adjacent phasors (phasors that correspond to adjacent slits) corresponding to a principal maximum?

Is there more than one such angle? If so, list all possible values.

B. Sketch three phasors that correspond to a minimum on the screen.

What is the angle between adjacent phasors? (*Hint:* It may be easier to determine this angle if the tail of each phasor is placed at the origin.)

C. Would twice the value you found above also correspond to a minimum? three times?

Verify your answer by sketching the phasors and determining the angle between adjacent phasors for each case. Place the tails at the origin.

D. Write out the first few values of $\Delta\varphi_{adj}$, the phase difference between *adjacent* slits, corresponding to minima for three slits.

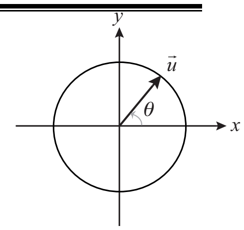
How many minima are between adjacent principle maxima for three slits? Is your answer consistent with your answers in an earlier tutorial, *Multiple-slit interference* tutorial?

E. Sketch phasors to represent minima for cases involving four, five, and six slits separated by equal distance.

Do you notice any patterns in how the phasors are distributed around a circle?

PHASORS WO 31

I. Use of rotating vectors to represent simple harmonic motion



Consider a *unit* vector $\vec{u}$ in the *xy*-plane as shown.

A. Write an expression for the *y*-component, u_y of $\vec{u}$ in terms of the angle θ shown in the figure.

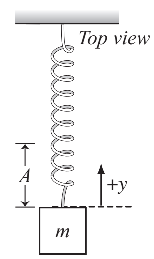
B. Suppose $\vec{u}$ is rotating counter-clockwise in the *xy*-plane with a constant angular frequency ω. Let $\vec{u}$ point entirely in the positive *x*-direction at $t = 0$.

Write an expression for u_y as a function of time.

Describe in words how u_y changes as t increases.

C. Now consider a block attached to an ideal spring that is moving with amplitude A about the equilibrium position on a frictionless surface, as shown at right. Assume at $t = 0$, the block is at its equilibrium position and moving in the $+y$ direction with angular frequency ω_s.

Write an expression for the displacement of the block as a function of time.



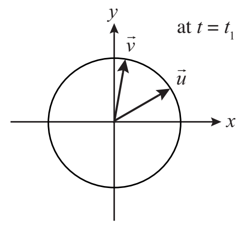
Describe how the vector $\vec{u}$ in uniform circular motion above can be used to represent one-dimensional simple harmonic motion.

Any sinusoidal function, such as $y(t) = \sin(\omega t)$ can be represented by the *y*-component of a rotating vector. (The other component of the rotating vector is ignored.) The rotating vector is sometimes called a *phasor*. By convention, phasors are typically treated as rotating counter-clockwise when the argument in the sinusoidal function is increasing. The argument of a sinusoidal function is defined as the *phase,* which can be represented by the angle between the phasor and the positive horizontal axis in a phasor diagram.

➪ Check your answers with a tutorial instructor.

Figure A.19: *Phasors* in-class tutorial worksheet administered in Autumn 2017 to PHYS 123 (six pages).

II. Addition of rotating vectors



A. The diagram at right shows $\vec{u}$ from section I and another unit vector, $\vec{v}$, at some time $t = t_1$. Suppose both vectors rotate counter-clockwise with the same angular frequency ω. The angle between $\vec{v}$ and $\vec{u}$ is $\delta\varphi$.

Write an expression for u_y at $t = t_1$. Recall that $\vec{u}$ points entirely in the positive x-direction at $t = 0$.

B. What is the *phase* of vector $\vec{v}$ at $t = t_1$? Explain.

Use your answers above to write an expression for v_y at $t = t_1$ as a sine function.

C. On the diagram, draw $\vec{s}$, the vector sum of $\vec{v}$ and $\vec{u}$ at $t = t_1$. Make sure both the direction and the magnitude are correct.

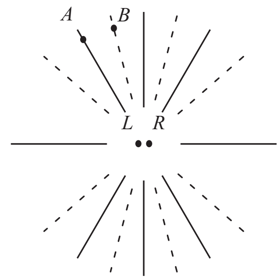
How does the vector $\vec{s}$ change over time? In particular, how does the angular frequency of $\vec{s}$ compared to the angular frequency of $\vec{u}$ and $\vec{v}$? Explain.

D. Do you think it would be appropriate to use phasors to represent the *superposition* of waves? Explain why or why not.

➪ Check your answers with a tutorial instructor.

III. Use of phasors to represent water waves

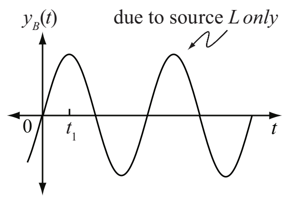
Consider two point sources of water waves, L and R. The diagram at right shows the nodal lines (dashed) and antinodal lines (solid) due to these two sources.



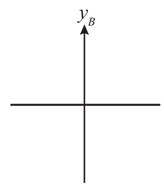
Top View

A. The graph at left below shows the displacement of the water surface *versus* time at point B due to source L *only*.

1. On the same graph, sketch the displacement over time at point B due to source R only. Explain.



Displacement vs. time at point B



Phasor diagram at point B at $t = t_1$

2. On the phasor diagram above, sketch and label a pair of phasors to represent the displacement at point B due to each of the two sources at $t = t_1$. (*Note:* t_1 is labeled on the graph.)

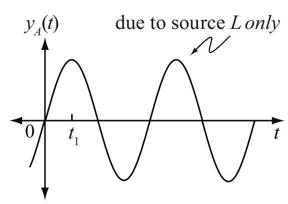
   How do the phasors at point B change over time? Explain.

   How can the motion of the water surface for point B be inferred from the phasor diagram?

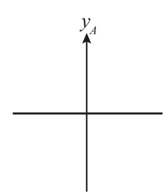
3. What is the phase difference between the phasors at $t = t_1$? Does it change over time? Explain.

B. Now consider point A in the top view diagram.

1. On the diagram at right, (1) sketch the displacement over time due to source R only, and (2) a pair of phasors to represent the displacement due to each source at $t = t_1$. Explain.



Displacement vs. time at point A



Phasor diagram at point A at $t = t_1$

2. What is the phase difference between the phasors at $t = t_1$?

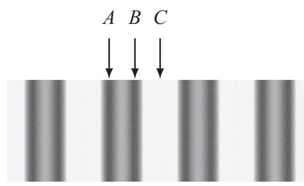
3. Describe how the motion of the water surface for point *A* can be inferred from the phasor diagram.

C. What patterns do you notice about the phasor diagrams that represent the water surface at any point on nodal lines and antidonal lines? Explain.

➪ Check your answer with a tutorial instructor.

IV. Phasors and multiple-slit interference

Suppose that coherent red light were incident on a mask with *three* equally spaced narrow slits S_1, S_2, and S_3. The diagram at right illustrates the pattern that appears on a distant screen. Point *C* marks the central principal maximum on the screen. Points *A* and *B* mark minima.



Three-slit pattern on screen

A. Sketch and label three phasors that could correspond to the central principal maximum at point *C*.

For your sketch above, what is $\delta\varphi_{adj}$, the phase difference between adjacent phasors (phasors that correspond to adjacent slits)? Explain.

Now consider a short time later. Sketch three phasors that could correspond to the central principal maximum.

A phasor diagram is sometimes presented without specifying the time instant. Explain why it is not always necessary.

B. Now consider a principal maximum other than the central principal maximum.

Do you expect the phasor diagram to be the same or different from the phasor diagram you drew in part A? Explain.

Do you expect $\delta\varphi_{adj}$ to be the same as or different from what it was in part A? Explain.

C. Consider point *B*, the first minimum to the left of point *C*.

Sketch and label three phasors that could correspond to this minimum. Sketch your diagram in two ways: (1) connect the phasors tail to tail, and (2) connect the phasors head to tail.

What is $\delta\varphi_{\text{adj}}$ for point *B?* Label it in the phasor diagram.

D. Now consider point *A*.

Sketch and label phasors that could correspond to this minimum in both ways. Label the value of $\delta\varphi_{\text{adj}}$ in the phasor diagram.

Describe how the phasor diagram for point *A* is *different* from the phasor diagram for point *B*.

E. On the three-slit pattern diagram on the previous page, label each of the minima with the corresponding values of $\delta\varphi_{\text{adj}}$ and δs_{adj}.

What are the units of $\delta\varphi_{\text{adj}}$? What are the units of δs_{adj}?

F. Now consider the secondary maximum between points *A* and *B*.

What is the value of $\delta\varphi_{\text{adj}}$ for this point? Briefly explain.

Sketch and label phasors that could correspond to this secondary maximum.

What is the ratio between the intensity of this point on the screen and the intensity of a principal maximum? (Recall that the intensity is proportional to the *square* of the maximum transverse displacement of a wave.) Explain.

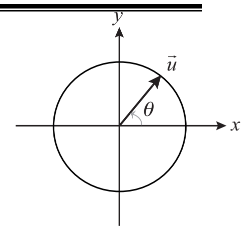
G. Sketch phasors to represent the *first* minima for cases involving four, five, and six slits separated by equal distance.

Four slits	*Five slits*	*Six slits*

Describe any pattern that you observe in how the phasors are distributed.

PHASORS WO 27

I. Use of rotating vectors to represent simple harmonic motion



Consider a *unit* vector $\vec{u}$ in the *xy*-plane as shown.

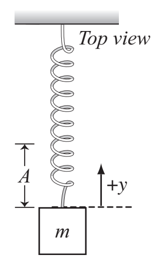
A. Write an expression for the *y*-component, u_y of $\vec{u}$ in terms of the angle θ shown in the figure.

B. Suppose $\vec{u}$ is rotating counter-clockwise in the *xy*-plane with a constant angular frequency ω_o. Let $\vec{u}$ point entirely in the positive *x*-direction at $t = 0$.

Write an expression for u_y as a function of time.

Describe in words how u_y changes as *t* increases.

C. Now consider a block attached to an ideal spring that is moving with amplitude *A* about the equilibrium position on a frictionless surface, as shown at right. Assume at $t = 0$, the block is at its equilibrium position and moving in the $+y$ direction with angular frequency ω_s.



Write an expression for the displacement of the block as a function of time.

Describe how the vector $\vec{u}$ in uniform circular motion above can be used to represent one-dimensional simple harmonic motion.

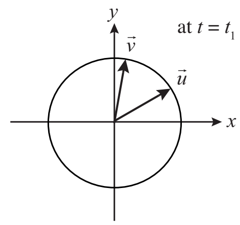
Any sinusoidal function, such as $y(t) = \sin(\omega t)$ can be represented by the *y*-component of a rotating vector. (The other component of the rotating vector is ignored.) The rotating vector is sometimes called a *phasor*. By convention, phasors are typically treated as rotating counter-clockwise when the argument in the sinusoidal function is increasing. The argument of a sinusoidal function is defined as the *phase,* which can be represented by the angle between the phasor and the positive horizontal axis in a phasor diagram.

➪ Check your answers with a tutorial instructor.

Figure A.20: *Phasors* in-class tutorial worksheet administered in Winter 2018 to PHYS 123 (six pages).

II. Addition of rotating vectors

A. The diagram at right shows $\vec{u}$ from section I and another unit vector, $\vec{v}$, at some time $t = t_1$. Suppose both vectors rotate counter-clockwise with the same angular frequency ω_o. The angle between $\vec{v}$ and $\vec{u}$ is $\delta\varphi$.



Write an expression for u_y at $t = t_1$. Recall that $\vec{u}$ points entirely in the positive x-direction at $t = 0$.

B. What is the *phase* of vector $\vec{v}$ at $t = t_1$? Explain.

Use your answers above to write an expression for v_y at $t = t_1$ as a sine function.

C. On the diagram, draw $\vec{s}$, the vector sum of $\vec{v}$ and $\vec{u}$ at $t = t_1$. Make sure both the direction and the magnitude are correct.

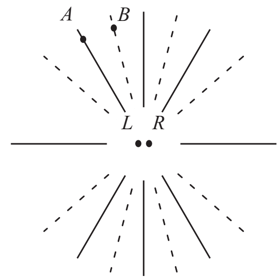
How does the vector $\vec{s}$ change over time? In particular, how does the angular frequency of $\vec{s}$ compared to the angular frequency of $\vec{u}$ and $\vec{v}$? Explain.

D. Phasors can be used to represent waves. Explain why this is appropriate.

Describe how to use phasors to represent interference of two waves.

➪ Check your answers with a tutorial instructor.

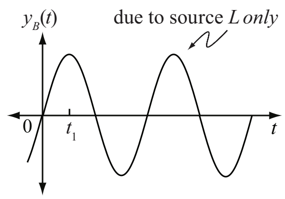
III. Use of phasors to represent water waves



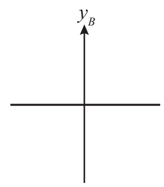
Consider two point sources of water waves, L and R. The diagram at right shows the nodal lines (dashed) and antinodal lines (solid) due to these two sources.

A. The graph at left below shows the displacement of the water surface *versus* time at point B due to source L *only*.

1. On the same graph, sketch the displacement over time at point B due to source R only. Explain.



Displacement vs. time at point B



Phasor diagram at point B at $t = t_1$

2. On the phasor diagram above, sketch and label a pair of phasors to represent the displacement at point B due to each of the two sources at $t = t_1$. (*Note:* t_1 is labeled on the graph.)

   How do the phasors at point B change over time? Explain.

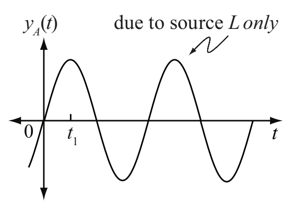
   How can the motion of the water surface for point B be inferred from the phasor diagram?

3. What is the phase difference between the phasors at $t = t_1$? Does it change over time? Explain.

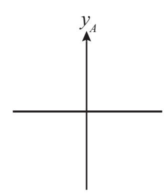
B. Now consider point A in the top view diagram.

1. On the diagram at right, (1) sketch the displacement over time due to source R only, and (2) a pair of phasors to represent the displacement due to each source at $t = t_1$. Explain.

2. What is the phase difference between the phasors at $t = t_1$?



Displacement vs. time at point A



Phasor diagram at point A at $t = t_1$

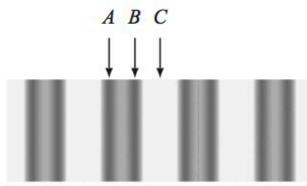
3. Describe how the motion of the water surface for point A can be inferred from the phasor diagram.

C. What patterns do you notice about the phasor diagrams that represent the water surface at any point on nodal lines and antinodal lines? Explain.

➪ Check your answer with a tutorial instructor.

IV. Phasors and multiple-slit interference

Suppose that coherent red light were incident on a mask with *three* equally spaced narrow slits S_1, S_2, and S_3. The diagram at right illustrates the pattern that appears on a distant screen. Point C marks the central principal maximum on the screen. Points A and B mark minima.



Three-slit pattern on screen

A. Sketch and label three phasors that could correspond to the central principal maximum at point C.

For your sketch above, what is $\delta\varphi_{adj}$, the phase difference between adjacent phasors (phasors that correspond to adjacent slits)? Explain.

Now consider a short time later. Sketch three phasors that could correspond to the central principal maximum.

A phasor diagram is sometimes presented without specifying the time instant. Explain why it is not always necessary.

B. Now consider a principal maximum other than the central principal maximum.

Do you expect the phasor diagram to be the same or different from the phasor diagram you drew in part A? Explain.

Do you expect $\delta\varphi_{adj}$ to be the same as or different from what it was in part A? Explain.

C. Consider point *B*, the first minimum to the left of point *C*.

Sketch and label three phasors that could correspond to this minimum. Sketch your diagram in two ways: (1) connect the phasors tail to tail, and (2) connect the phasors head to tail.

What is $\delta\varphi_{\text{adj}}$ for point *B?* Label it in the phasor diagram.

D. Now consider point *A*.

Sketch and label phasors that could correspond to this minimum in both ways. Label the value of $\delta\varphi_{\text{adj}}$ in the phasor diagram.

Describe how the phasor diagram for point *A* is *different* from the phasor diagram for point *B*.

E. On the three-slit pattern diagram on the previous page, label each of the minima with the corresponding values of $\delta\varphi_{\text{adj}}$ and δs_{adj}.

What are the units of $\delta\varphi_{\text{adj}}$? What are the units of δs_{adj}?

F. Now consider the secondary maximum between points *A* and *B*.

What is the value of $\delta\varphi_{\text{adj}}$ for this point? Briefly explain.

Sketch and label phasors that could correspond to this secondary maximum.

What is the ratio between the intensity of this point on the screen and the intensity of a principal maximum? (Recall that the intensity is proportional to the *square* of the maximum transverse displacement of a wave.) Explain.

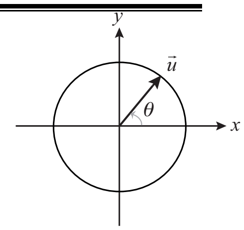
G. Sketch phasors to represent the *first* minima for cases involving four, five, and six slits separated by equal distance.

Four slits	*Five slits*	*Six slits*

Describe any pattern that you observe in how the phasors are distributed.

PHASORS WO 27

I. Use of rotating vectors to represent simple harmonic motion



Consider a *unit* vector $\vec{u}$ in the *xy*-plane as shown.

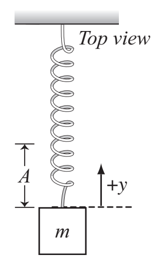
A. Write an expression for the *y*-component, u_y of $\vec{u}$ in terms of the angle θ shown in the figure.

B. Suppose $\vec{u}$ is rotating counter-clockwise in the *xy*-plane with a constant angular frequency ω_o. Let $\vec{u}$ point entirely in the positive *x*-direction at $t = 0$.

Write an expression for u_y as a function of time.

Describe in words how u_y changes as *t* increases.

C. Now consider a block attached to an ideal spring that is moving with amplitude *A* about the equilibrium position on a frictionless surface, as shown at right. Assume at $t = 0$, the block is at its equilibrium position and moving in the $+y$ direction with angular frequency ω_o.



Write an expression for the displacement of the block as a function of time.

Describe how the vector $\vec{u}$ in uniform circular motion above can be used to represent one-dimensional simple harmonic motion.

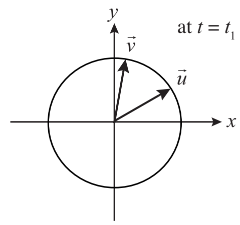
Any sinusoidal function, such as $y(t) = \sin(\omega t)$ can be represented by the *y*-component of a rotating vector. (The other component of the rotating vector is ignored.) The rotating vector is sometimes called a *phasor*. By convention, phasors are typically treated as rotating counter-clockwise when the argument in the sinusoidal function is increasing. The argument of a sinusoidal function is defined as the *phase,* which can be represented by the angle between the phasor and the positive horizontal axis in a phasor diagram.

 Check your answers with a tutorial instructor.

Figure A.21: *Phasors* in-class tutorial worksheet administered in Spring 2018 to PHYS 123 (six pages).

II. Addition of rotating vectors

A. The diagram at right shows $\vec{u}$ from section I and another unit vector, $\vec{v}$, at some time $t = t_1$. Suppose both vectors rotate counter-clockwise with the same angular frequency ω_0. The angle between $\vec{v}$ and $\vec{u}$ is $\delta\varphi$.



Write an expression for u_y at $t = t_1$. Recall that $\vec{u}$ points entirely in the positive x-direction at $t = 0$.

B. What is the *phase* of vector $\vec{v}$ at $t = t_1$? Explain.

Use your answers above to write an expression for v_y at $t = t_1$ as a sine function.

C. On the diagram, draw $\vec{s}$, the vector sum of $\vec{v}$ and $\vec{u}$ at $t = t_1$. Make sure both the direction and the magnitude are correct.

How does the vector $\vec{s}$ change over time? In particular, how does the angular frequency of $\vec{s}$ compared to the angular frequency of $\vec{u}$ and $\vec{v}$? Explain.

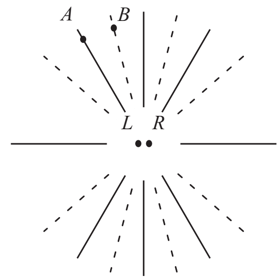
D. Phasors can be used to represent waves. Explain why this is appropriate.

Describe how to use phasors to represent interference of two waves.

☐ Check your answers with a tutorial instructor.

III. Use of phasors to represent water waves

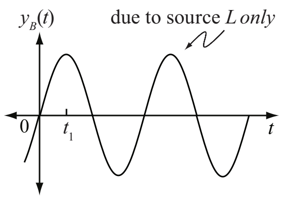
Consider two point sources of water waves, *L* and *R*. The diagram at right shows the nodal lines (dashed) and antinodal lines (solid) due to these two sources.



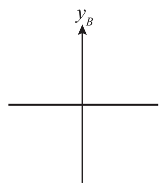
Top View

A. The graph at left below shows the displacement of the water surface *versus* time at point *B* due to source *L only*.

1. On the same graph, sketch the displacement over time at point *B* due to source *R* only. Explain.



Displacement vs. time at point B



Phasor diagram at point B at $t = t_1$

2. On the phasor diagram above, sketch and label a pair of phasors to represent the displacement at point *B* due to each of the two sources at $t = t_1$. (*Note:* t_1 is labeled on the displacement vs. time graph.)

   How do the phasors at point *B* change over time? Explain.

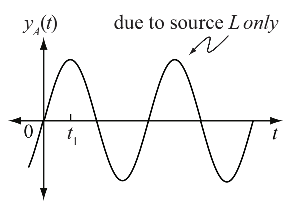
   How can the motion of the water surface for point *B* be inferred from the phasor diagram?

3. What is the phase difference between the phasors at $t = t_1$? Does it change over time? Explain.

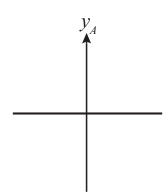
B. Now consider point *A* in the top view diagram.

1. On the diagram at right, (1) sketch the displacement over time due to source *R* only, and (2) a pair of phasors to represent the displacement due to each source at $t = t_1$. Explain.

2. What is the phase difference between the phasors at $t = t_1$?



Displacement vs. time at point A



Phasor diagram at point A at $t = t_1$

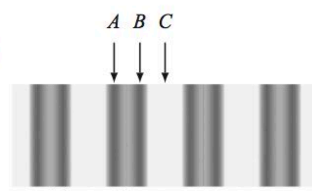
3. Describe how the motion of the water surface for point A can be inferred from the phasor diagram.

C. What patterns do you notice about the phasor diagrams that represent the water surface at any point on nodal lines and antinodal lines? Explain.

☐ Check your answer with a tutorial instructor.

IV. Phasors and multiple-slit interference

Suppose that coherent red light were incident on a mask with *three* equally spaced narrow slits S_1, S_2, and S_3. The diagram at right illustrates the pattern that appears on a distant screen. Point C marks the central principal maximum on the screen. Points A and B mark minima.



Three-slit pattern on screen

A. Sketch and label three phasors that could correspond to the central principal maximum at point C.

For your sketch above, what is $\delta\varphi_{adj}$, the phase difference between adjacent phasors (phasors that correspond to adjacent slits)? Explain.

Now consider a short time later. Sketch three phasors that could correspond to the central principal maximum.

A phasor diagram is sometimes presented without specifying the time instant. Explain why it is not always necessary.

B. Now consider a principal maximum other than the central principal maximum.

Do you expect the phasor diagram to be the same or different from the phasor diagram you drew in part A? Explain.

Do you expect $\delta\varphi_{adj}$ to be the same as or different from what it was in part A? Explain.

C. Consider point *B*, the first minimum to the left of point *C*.

Sketch and label three phasors that could correspond to this minimum. Sketch your diagram in two ways: (1) connect the phasors tail to tail, and (2) connect the phasors head to tail.

What is $\delta\varphi_{adj}$ for point *B?* Label it in the phasor diagram.

D. Now consider point *A*.

Sketch and label phasors that could correspond to this minimum in both ways. Label the value of $\delta\varphi_{adj}$ in the phasor diagram.

Describe how the phasor diagram for point *A* is *different* from the phasor diagram for point *B*.

E. On the three-slit pattern diagram on the previous page, label each of the minima with the corresponding values of $\delta\varphi_{adj}$ and δs_{adj}.

What are the units of $\delta\varphi_{adj}$? What are the units of δs_{adj}?

F. Now consider the secondary maximum between points *A* and *B*.

What is the value of $\delta\varphi_{adj}$ for this point? Briefly explain.

Sketch and label phasors that could correspond to this secondary maximum.

What is the ratio between the intensity of this point on the screen and the intensity of a principal maximum? (Recall that the intensity is proportional to the *square* of the maximum transverse displacement of a wave.) Explain.

G. Sketch phasors to represent the *first* minima for cases involving four, five, and six slits separated by equal distance.

Four slits	*Five slits*	*Six slits*

Describe any pattern that you observe in how the phasors are distributed.

APPENDIX B

Tutorial Homework

This appendix consists of copies of the tutorial homework discussed in this dissertation.

HOMEWORK: PROBABILITY AMPLITUDE Name ____________________ QM HW-1

1. Consider a particle in the quantum mechanical infinite square well from $x = 0$ to $x = a$. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$. Let the system's position eigenstate and eigenvalue be given by $|x\rangle$ and x, respectively, where $0 \le x \le a$.

   a. Consider the operator given by the expression $|\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$?

      ii. Give a physical interpretation for the inner product that results from this operator.

   b. Consider the operator given by the expression $\sum_n |\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$?

      ii. Compare the result with the expression you wrote in question A of section I of the tutorial. Is the state that results from the action of the operator *the same as* or *different from* the initial state $|\varphi_A\rangle$? Explain.

   c. Consider the operator given by the expression $\int_0^a dx|x\rangle\langle x|$.

      i. Write an expression for this operator acting on the state $|\varphi_A\rangle$? Explain.

      ii. Compare the result with the expression you wrote in question F of section II of the tutorial. Is the state that results from the action of the operator *the same as* or *different from* the initial state $|\varphi_A\rangle$? Explain.

Figure B.1: *Probability Amplitude* tutorial homework administered in Autumn 2015 to PHYS 324 (four pages).

d. Let $|p\rangle$ and p represent the eigenstates and eigenvalues, respectively, of the momentum operator.

   i. Describe how to find the probability that a measurement of momentum results in a value between $p = k_1$ and $p = k_2$.

   ii. Write an expression for the state $|\varphi_A\rangle$, in terms of the eigenstates of the momentum operator. Explain.

e. Consider a hypothetical operator, $\hat{Y}$.

   i. Suppose the eigenvalues of $\hat{Y}$ are discrete. Let the eigenstates and eigenvalues be given by $|y_n\rangle$ and y_n, respectively, for $n = 1, 2, 3, \ldots$. Describe how to find the probability of measuring y_n.

   ii. Now suppose the eigenvalues of operator $\hat{Y}$ are continuous. Let the eigenstates and eigenvalues be given by $|y\rangle$ and y, respectively, where y is a continuous variable. Describe how to find the probability of measuring eigenvalue y in the region between $y = b$ and $y = c$.

2. The infinite square well of width a has energy eigenfunctions $\psi_n(x) = \sqrt{\frac{2}{a}}\sin\left(\frac{n\pi x}{a}\right)$.

Homework: probability amplitude Name ____________ QM HW-3

Suppose that particle B has the initial wave function defined at right. (*Note:* The wave function is zero outside of the well.)

$$\varphi(x,t=0)=\sqrt{\frac{2}{a}}\begin{cases}\sin\left(\frac{2\pi x}{a}\right) & 0\le x\le \frac{a}{2}\\ -\sin\left(\frac{2\pi x}{a}\right) & \frac{a}{2}\le x\le a\end{cases}$$

a. Sketch the wave function for this particle in the space below. (*Hint:* Is the wave function *positive* or *negative* on the right half of the well?)

b. Sketch the probability density for this particle in the space below.

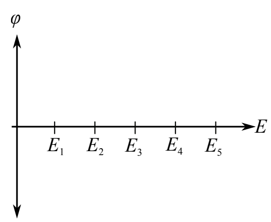
c. Explain why the normalization constant for this particle's wave function is the same as the normalization constant for the first excited-state wave function.

d. Predict whether the probability that this particle is measured to have energy E_1 is *greater than*, *less than*, or *equal to* the probability that this particle is measured to have energy E_2. Explain your reasoning.

e. *Predict* whether the probability that this particle is measured to have energy E_1 is *greater than*, *less than*, or *equal to* the probability that this particle is measured to have energy E_3. Explain your reasoning.

f. Determine the first five coefficients in the energy basis. (You may wish to use computer software to carry out the integration.)

g. Graph this state on the axes below using the energy representation.



h. Optional (extra credit): Determine the momentum-space wave function for this particle and sketch a graph of this function in the space below or using a computer program.

HOMEWORK: PROBABILITY AMPLITUDE AND INTERFERENCE Name ____________________ QM HW-1

1. Suppose that $|\varphi\rangle$ represents the state of a particle. Which of the expressions below is an example of a probability amplitude? Explain.

   a. $\langle p|\varphi\rangle$

   b. $\hat{H}|\varphi\rangle$

   c. $|\varphi(x)|^2$

   d. $\int x\varphi(x)dx$

2. Consider a particle in a quantum mechanical infinite square well that extends from $x = 0$ to $x = a$. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$. Let the system's position eigenstate and eigenvalue be given by $|x\rangle$ and x, respectively, where $0 \le x \le a$.

   a. Consider the operator given by the expression $|\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$.

      ii. What part in your expression above represents a probability amplitude? Give an interpretation for this probability amplitude.

   b. Consider the operator given by the expression $\sum_n |\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$.

      ii. Does this operator change the state? (*Hint:* Compare the result with the expression you wrote in question A of section I of the tutorial.)

      Give an interpretation for this operator. Explain.

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Figure B.2: *Probability Amplitude* tutorial homework administered in Autumn 2016 to PHYS 324 (three pages).

c. Consider the operator given by the expression $\int_0^a dx|x\rangle\langle x|$.

i. Write an expression for this operator acting on the state $|\varphi_A\rangle$.

ii. Does this operator change the state? (*Hint:* Compare the result with the expression you wrote in question C of section II of the tutorial.)

Give an interpretation for this operator. Explain.

iii. How are the operators $\int_0^a dx|x\rangle\langle x|$ and $\sum_n |\psi_n\rangle\langle\psi_n|$ similar?

d. Let $|p\rangle$ and p represent the eigenstates and eigenvalues, respectively, of the momentum operator.

i. Use the momentum eigenstates to write an expression for the probability that a particle is measured to be moving to the left.

ii. Write an expression for the state $|\varphi_A\rangle$, in terms of the eigenstates of the momentum operator. Explain.

Describe how this expression is similar to the expression for the same state in terms of the eigenstates of the position operator. Explain.

e. Consider a hypothetical operator, $\hat{Y}$.

i. Suppose the eigenvalues of $\hat{Y}$ are discrete. Let the eigenstates and eigenvalues be given by $|y_n\rangle$ and y_n, respectively, for $n = 1, 2, 3, \ldots$. Write an expression for the probability of measuring y_n. Briefly explain.

ii. Now suppose the eigenvalues of operator $\hat{Y}$ are continuous. Let the eigenstates and eigenvalues be given by $|y\rangle$ and y, respectively, where y is a continuous variable. Write an expression for the probability of measuring the eigenvalue y in the region between $y = b$ and $y = c$. Briefly explain.

HOMEWORK: PROBABILITY AMPLITUDE AND INTERFERENCE Name ____________________ QM HW-1

1. Consider a particle in a quantum mechanical infinite square well that extends from $x = 0$ to $x = a$. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$. Let the system's position eigenstate and eigenvalue be given by $|x\rangle$ and x, respectively, where $0 \le x \le a$.

   a. Consider the operator given by the expression $|\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$.

      ii. What part in your expression above represents a probability amplitude? Give an interpretation for this probability amplitude.

   b. Consider the operator given by the expression $\sum_n |\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for this operator acting on the state, $|\varphi_A\rangle$.

      ii. Give an interpretation for the operation when this operator acts on a state. Explain.

   c. Consider the operator given by the expression $\int_0^a dx|x\rangle\langle x|$.

      i. Write an expression for this operator acting on the state $|\varphi_A\rangle$.

      ii. Give an interpretation for the operation when this operator acts on a state. Explain. (*Hint:* Compare the result with the expression you wrote in question G of section III of the tutorial.)

Figure B.3: *Probability Amplitude* tutorial homework administered in Summer 2017 to PHYS 324 (three pages).

iii. How are the operators $\int_0^a dx|x\rangle\langle x|$ and $\sum_n |\psi_n\rangle\langle\psi_n|$ similar?

iv. Use operator $\int_0^a dx|x\rangle\langle x|$ to rewrite each of the inner products below as integrals. Show your work.

$\langle\varphi_A|\varphi_A\rangle$

$\langle\psi_n|\varphi_A\rangle$

$\langle x|\varphi_A\rangle$ (*Hint:* Make sure the result of your integral is consistent with the definition of this inner product.)

Based on your integral above, what is the eigenfunction for position eigenstate $|x\rangle$?

d. Let $|p\rangle$ and p represent the eigenstates and eigenvalues, respectively, of the *momentum* operator.

i. Use the momentum eigenstates to write an expression for the probability that a particle is measured to be moving to the left.

ii. Write an expression for the state $|\varphi_A\rangle$, in terms of the eigenstates of the momentum operator. Explain.

Describe how this expression is similar to the expression for the same state in terms of the eigenstates of the position operator. Explain.

e. Consider a hypothetical operator, $\hat{Y}$.

i. Suppose the eigenvalues of $\hat{Y}$ are discrete. Let the eigenstates and eigenvalues be given by $|y_n\rangle$ and y_n, respectively, for $n = 1, 2, 3, \ldots$. Write an expression for the probability of measuring y_n. Briefly explain.

ii. Now suppose the eigenvalues of operator $\hat{Y}$ are continuous. Let the eigenstates and eigenvalues be given by $|y\rangle$ and y, respectively, where y is a continuous variable. Write an expression for the probability of measuring the eigenvalue y in the region between $y = b$ and $y = c$. Briefly explain.

HOMEWORK: PROBABILITY AMPLITUDE

Name ____________________ QM

Section: HW-1

1. Suppose that $|\varphi\rangle$ represents the state of a particle. Let $|\psi_n\rangle$ and $|x\rangle$ represent the energy eigenstate and the position eigenstate, respectively. Write an expression for the result of each of the following inner products. Explain how obtained your results.

   a. $\langle\psi_n|\varphi\rangle$

   b. $\langle\varphi|\varphi\rangle$

   c. $\langle\psi_n|\psi_m\rangle$

   d. $\langle x|\varphi\rangle$

   e. $\langle x|x'\rangle$

2. Consider a particle in a quantum mechanical infinite square well that extends from $x = 0$ to $x = a$. Let the state of the particle be given by $|\varphi_A\rangle$ and let the system's energy eigenstates and eigenvalues be given by $|\psi_n\rangle$ and E_n, respectively, for $n = 1, 2, 3, \ldots$. Let the system's position eigenstates and eigenvalues be given by $|x\rangle$ and x, respectively, where $0 \le x \le a$.

   a. Consider the operator given by the expression $|\psi_n\rangle\langle\psi_n|$.

      i. Write an expression for the result of this operator acting on the state $|\varphi_A\rangle$.

      ii. Give an interpretation for this operator.

   b. Consider the operator given by the expression $\sum_n |\psi_n\rangle\langle\psi_n|$.

      Give an interpretation for the operation when it acts on a state. Explain.

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Figure B.4: *Probability Amplitude* tutorial homework administered in Autumn 2017 to PHYS 324 (three pages).

c. Consider the operator given by the expression $\int_0^a dx|x\rangle\langle x|$.

i. Write an expression for the result of this operator acting on the state $|\varphi_A\rangle$. Simplify this expression.

ii. Give an interpretation for the operator when it acts on a state. Explain. (*Hint:* Compare the result with the expression you wrote in question G of section III of the tutorial.)

iii. How are the operators $\int_0^a dx|x\rangle\langle x|$ and $\sum_n|\psi_n\rangle\langle\psi_n|$ similar?

How are they different?

iv. Use operator $\int_0^a dx|x\rangle\langle x|$ to show how the following inner products below can be written as integrals and then evaluate the integrals. Show your work.

$\langle\varphi_A|\varphi_A\rangle$

$\langle\psi_n|\varphi_A\rangle$

$\langle x'|\varphi_A\rangle$

Are your results above consistent with your expressions in part 1? Resolve any inconsistencies.

d. Let $|p\rangle$ and p represent the eigenstates and eigenvalues, respectively, of the *momentum* operator.

i. Write an expression for the state $|\varphi_A\rangle$ in terms of the eigenstates of the momentum operator. Explain.

Describe how this expression is similar to the expression for the same state in terms of the eigenstates of the position operator. Explain.

ii. Write an expression for the probability that a particle is measured to be moving to the left. If you are using any quantities that have not been defined in part 2, rewrite those quantities in terms of the quantities that have been defined.

e. Consider a hypothetical operator, $\hat{Y}$.

i. Suppose the eigenvalues of $\hat{Y}$ are discrete. Let the eigenstates and eigenvalues be given by $|y_n\rangle$ and y_n, respectively, for $n = 1, 2, 3, \ldots$. Write an expression for the probability of measuring y_n. Briefly explain.

ii. Now suppose the eigenvalues of operator $\hat{Y}$ are continuous. Let the eigenstates and eigenvalues be given by $|y\rangle$ and y, respectively, where y is a continuous variable. Write an expression for the probability of measuring the eigenvalue y in the region between $y = b$ and $y = c$. Briefly explain.

HOMEWORK: QUANTUM INTERFERENCE WITH SPIN STATES

Name ______________________

QM HW-1

1. Consider a spin-1/2 particle described by the state $|\psi\rangle = c_1|+\rangle_z + c_2|-\rangle_z$. Show that whether the result of interference depends on the overall phase or the relative phase using the following steps.

   a. Rewrite each coefficient as a positive real number multiplied by a complex exponential.

   b. Calculate the probability that the particle is measured to have spin up in the *x* direction.

   c. Does the probability depend on the relative phase between the two basis states $|+\rangle_z$ and $|-\rangle_z$? Explain.

   d. Determine the largest and smallest values for the probability (in terms of the magnitudes of c_1 and c_2) and the corresponding conditions.

   e. Describe how quantum interference is analogous to interference with classical waves.

   f. Suppose the particle is instead described by the state $|\phi\rangle = c_1 e^{i\theta}|+\rangle_z + c_2 e^{i\theta}|-\rangle_z$. Would the probability that the particle is measured to have spin up in the *x* direction be *the same* or *different than* your answer in part b? Explain or show your work.

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Figure B.5: *Quantum Interference with Spin States* tutorial homework administered in Winter 2017 to PHYS 225 (one page).

HOMEWORK: PHASORS

Name:______________________________

Section:

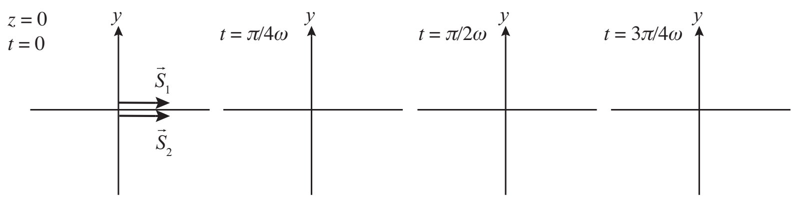
PO
HW–1

1. Consider two waves S_1 and S_2 with the same amplitude and wavelength traveling in *opposite* directions on a spring. Suppose that the transverse displacement of the spring due to wave S_1 can be written as $y_1(z, t) = \sin(kz + \omega t)$, where k and ω are constants.

   a. Give a physical interpretation to the constants k and ω.

   b. Write an expression for the transverse displacement of the spring due to wave S_2 in terms of position z and time t. Assume the waves are in phase at $t = 0$.

   c. The first diagram below uses phasors to represent the transverse displacement of the spring due to *each* wave at $z = 0$ and $t = 0$. Sketch the phasors at the same position ($z = 0$) for the indicated instants in time ($t = \frac{\pi}{4\omega}$, $t = \frac{\pi}{2\omega}$, and $t = \frac{3\pi}{4\omega}$).



   d. Describe how the *net* transverse displacement of the spring at position $z = 0$ changes in time. Explain. Justify your answer using the phasor diagrams above.

   e. In the diagrams below, sketch the phasors at position $z = \frac{\pi}{2k}$ for the indicated instants in time ($t = 0$, $t = \frac{\pi}{4\omega}$, $t = \frac{\pi}{2\omega}$, and $t = \frac{3\pi}{4\omega}$). Label each phasor.

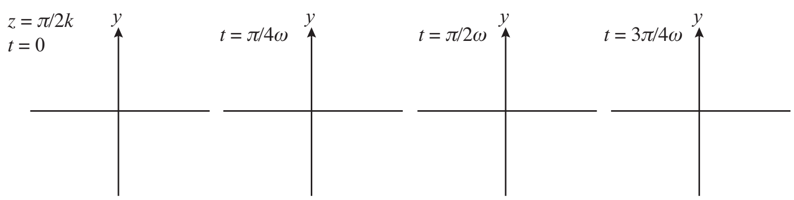
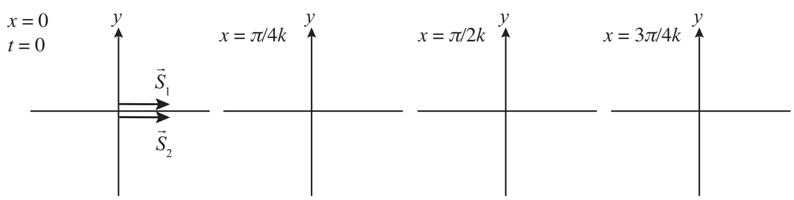


Figure B.6: *Phasors* tutorial homework administered in Summer 2016 to PHYS 123 (three pages).

f. Describe how the *net* transverse displacement of the spring at position $z = \frac{\pi}{2k}$ changes in time. Explain. Justify your answer using the phasor diagrams above.

g. The first diagram below uses phasors to represent the transverse displacement of the spring due to *each* wave at $z = 0$ and $t = 0$. Sketch the phasors at the *same* time ($t = 0$) for the indicated *positions* ($x = \frac{\pi}{4k}$, $x = \frac{\pi}{2k}$, and $x = \frac{3\pi}{4k}$).



h. Describe how the *net* transverse displacement of the spring at time $t = 0$ changes with *position*. Explain. Justify your answer using the phasor diagrams above.

i. The pattern of displacement formed by S_1 and S_2 is an example of a *standing wave*. Explain why this name is appropriate.

Homework: Phasors Name ______________________________ PO HW–3

2. Consider three-slit interference.

   a. How many minima are there between two adjacent principal maxima? Write out the values of phase difference $\Delta\varphi_{adj}$ for the minima.

   b. Suppose the phase difference between adjacent slits for the secondary maximum is half-way between values you found above. Sketch phasors to represent the three waves at this point on the screen and label the angles between the phasors.

   c. Use vector addition to determine the sum of the phasors.

   What is the magnitude of the resultant phasor? How does the magnitude compare to the magnitude of a single phasor?

   What does this tell you about the intensity of this point on the screen?

   d. Briefly explain why it is reasonable that $\Delta\varphi_{adj}$ for the secondary maximum is half-way between the values of $\Delta\varphi_{adj}$ for the minima.

HOMEWORK: PHASORS

Name:____________________________
Section:

WO
HW–23

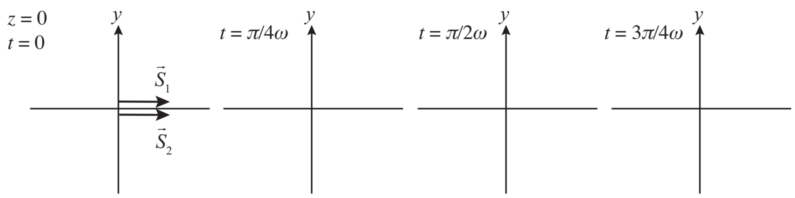
1. Consider two waves S_1 and S_2 with the same amplitude and wavelength traveling in *opposite* directions from the opposite ends of a spring. Suppose that the transverse displacement of the spring due to wave S_1 can be written as $y_1(z, t) = \sin(kz + \omega t)$, where k and ω are constants.

   a. Give a physical interpretation to the constants k and ω. For example, the physical interpretation of velocity for an object in uniform motion can be described as the displacement of the object that occurs in each unit of time.

   b. Write an expression for the transverse displacement of the spring due to wave S_2 in terms of position z and time t. Assume the waves are in phase at $t = 0$.

   c. The first diagram below uses phasors $\vec{S}_1$ and $\vec{S}_2$ to represent the transverse displacement of the spring due to *each* wave at $z = 0$ and $t = 0$.

   Sketch the phasors at the same position ($z = 0$) for the indicated instants in time *($t = \pi/4\omega$, $t = \pi/2\omega$,* and *$t = 3\pi/4\omega$).*



   d. Describe how the *net* transverse displacement of the spring at position $z = 0$ changes in time. Explain. Justify your answer using the phasor diagrams above.

   e. In the diagrams below, sketch the phasors at position $z = \pi/2k$ for the indicated instants in time *($t = 0$, $t = \pi/4\omega$, $t = \pi/2\omega$,* and *$t = 3\pi/4\omega$).* Label each phasor.

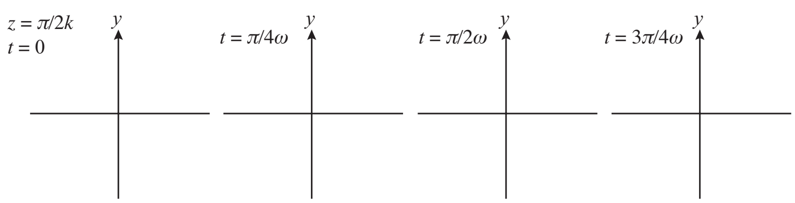
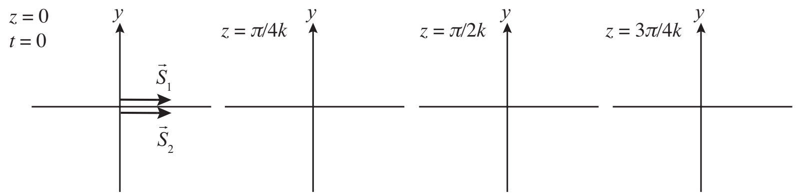


Figure B.7: *Phasors* tutorial homework administered in Autumn 2017 to PHYS 123 (four pages).

f. Describe how the *net* transverse displacement of the spring at position $z = \pi/2k$ changes in time. Explain. Justify your answer using the phasor diagrams above.

g. The first diagram below uses phasors to represent the transverse displacement of the spring due to *each* wave at $z = 0$ and $t = 0$. Sketch the phasors at the *same* time ($t = 0$) for the indicated *positions* ($z = \pi/4k$, $z = \pi/2k$ and $z = 3\pi/4k$).



h. Describe how the *net* transverse displacement of the spring at time $t = 0$ changes with *position*. Explain. Justify your answer using the phasor diagrams above.

i. The pattern of displacement formed by S_1 and S_2 is an example of a *standing wave*. Explain why this name is appropriate.

2. Suppose that coherent red light were incident on a mask with four equally spaced narrow slits S_1, S_2, S_3, and S_4. Recall that there are three minima between adjacent principal maxima in this case.

   a. Suppose that each of the minima is represented by a phasor diagram.

   Predict whether all the phasor diagrams would look the same if the phasors were *not* labeled in any way. Explain.

   b. Consider each of the minima between the central principal maximum and the first maximum to the right of the center.

      i. What are the values of $\delta\varphi_{\text{adj}}$ for those minima?

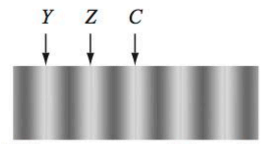
      ii. Sketch a phasor diagram for each minimum above. Label the corresponding slits (S_1, S_2, S_3 or S_4) and $\delta\varphi_{\text{adj}}$ in each phasor diagram as was done in tutorial.

First minimum	*Second minimum*	*Third minimum*

   c. Are the phasor diagrams you drew above all identical to one another? Is your answer consistent with your prediction in part b?

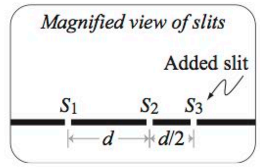
   d. Suppose that coherent red light were incident on a mask of N slits. In general, do you expect the phasor diagrams for all the minima to be identical? Explain.

3. The diagram at right illustrates the pattern that appears on a distant screen when coherent red light is incident on a mask with two identical, very narrow slits, S_1 and S_2. The two slits are separated by a distance d. Point C represents the center of the screen.



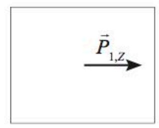
Pattern on screen with two slits

Suppose that a third slit, S_3, of equal width were added. It is a distance $d/2$ to the right of slit S_2 as shown in the figure below the pattern. (The new pattern on the screen is not shown.)



e. Consider point Z on the screen. Let phasor $\vec{P}_{1,Z}$ (see figure below right) represent the light at this point from slit S_1.

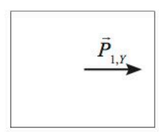
i. Sketch and label the phasor that represents the light at point Z from slit S_2. Explain your reasoning.



ii. Sketch and label the phasor that represents the light at point Z from slit S_3. Explain your reasoning.

iii. What is the intensity at point Z when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for three slits. Explain your reasoning.

f. Now consider point Y on the screen. Let phasor $\vec{P}_{1,Y}$ (see figure at right) represent the light at point Y from slit S_1.



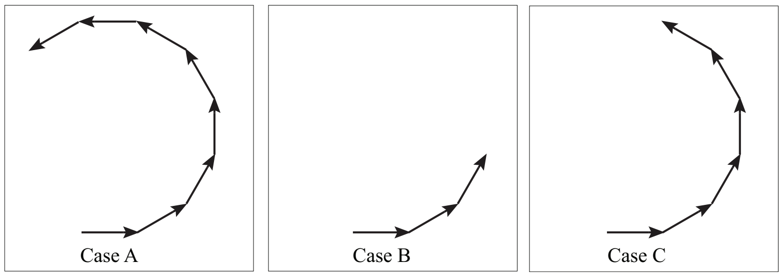
i. Sketch and label the phasors that represent the light at point Y from slits S_2 and S_3. Explain your reasoning.

ii. What is the intensity at point Y when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for three slits. Explain your reasoning.

PHASORS

Name:____________________________ WO
Section: HW–15

1. Coherent red light is incident on a mask with an unknown number of equally spaced narrow slits. Each of the phasor diagrams below corresponds to the same location on a distant screen. The only difference among the three physical situations is the number of slits.



a. How many slits are there in each case? Explain.

b. In case A, $\delta\varphi_{adj}$ is $\pi/6$. What is the corresponding value of δs_{adj} in terms of wavelength? Explain.

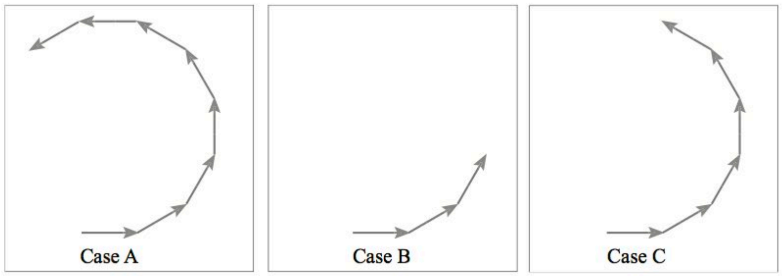
Would $\delta\varphi_{adj}$ and δs_{adj} be the same for cases B and C as it is for case A? Explain.

c. Rank the cases according to the intensity, from largest to smallest. Explain.

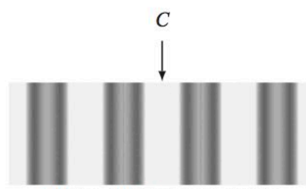
d. Based on your answers above, does the intensity always increase when the number of slits increases? Explain.

Figure B.8: *Phasors* tutorial homework administered in Winter 2018 to PHYS 123 (four pages).

e. Now suppose in each case one of the outermost slits is covered. In each case, would the intensity *increase, decrease,* or *stay the same?* Explain. As part of your explanation, sketch a possible phasor diagram for each case. The original phasor diagrams are duplicated for reference.

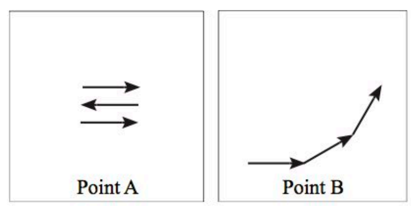


2. Coherent red light is incident on a mask with three equally spaced narrow slits. The interference pattern on the screen is shown at right. Recall that in this tutorial, the $\delta\varphi_{adj}$ of 3π and π are not equivalent.

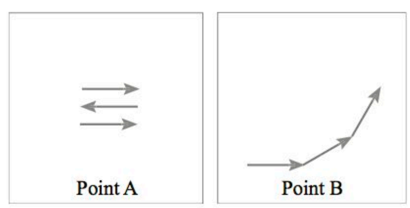


Three-slit pattern on screen

   a. Consider the phasor diagrams below right corresponding to points A and B on the screen. The $\delta\varphi_{adj}$ for points A and B are 3π and $\pi/6$, respectively. Mark the locations of points A and B on the interference pattern and explain how you determined the locations.



   b. Suppose a fourth slit is added to the right of the original three slits such that distances between adjacent slits are the same as before. In each case, would the intensity *increase, decrease,* or *stay the same?* Explain. As part of your explanation, sketch a possible phasor diagram for each case. The original phasor diagrams are duplicated for reference.



3. Consider each of the minima between the central principal maximum and the first maximum to the right of the center. Suppose that coherent red light were incident on a mask with four equally spaced narrow slits S_1, S_2, S_3, and S_4. Recall that there are three minima between adjacent principal maxima in this case.

   a. Suppose that each of the minima is represented by a phasor diagram.

   Predict whether all the phasor diagrams would look the same if the phasors were *not* labeled in any way. Explain.

   b. Consider each of the minima between the central principal maximum and the first maximum to the right of the center.

   i. What are the values of $\delta\varphi_{adj}$ for those minima?

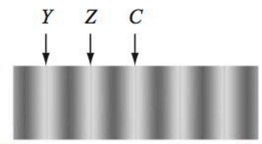
   ii. Use your answers above to sketch a phasor diagram for each minimum.

First minimum	*Second minimum*	*Third minimum*

   iii. Label the corresponding slits (S_1, S_2, S_3 or S_4) and $\delta\varphi_{adj}$ in each phasor diagram as was done in tutorial.

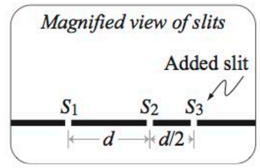
   The phasor diagrams you drew above should not all be identical. Check the value of $\delta\varphi_{adj}$ for each phasor diagram and resolve any inconsistencies.

4. The diagram at right illustrates the pattern that appears on a distant screen when coherent red light is incident on a mask with two identical, very narrow slits, S_1 and S_2. The two slits are separated by a distance d. Point C represents the center of the screen.



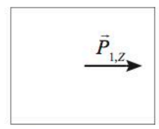
Pattern on screen with two slits

Suppose that a third slit, S_3, of equal width were added. It is a distance $d/2$ to the right of slit S_2 as shown in the figure below the pattern. (The new pattern on the screen is not shown.)



a. Consider point Z on the screen. Let phasor $\vec{P}_{1,Z}$ (see figure below right) represent the light at this point from slit S_1.

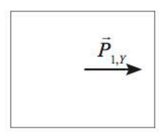
i. In the figure, sketch $\vec{P}_{2,Z}$, the phasor that represents the light at point Z from slit S_2. Explain how you determined the phase difference between $\vec{P}_{2,Z}$ and $\vec{P}_{1,Z}$.



ii. Sketch $\vec{P}_{3,Z}$, the phasor that represents the light at point Z from slit S_3. Explain how you determined the phase difference between $\vec{P}_{3,Z}$ and $\vec{P}_{2,Z}$.

iii. What is the intensity at point Z when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for *three* slits. Explain your reasoning.

b. Now consider point Y on the screen. Let phasor $\vec{P}_{1,Y}$ (see figure at right) represent the light at point Y from slit S_1.



i. In the figure, sketch $\vec{P}_{2,Y}$ and $\vec{P}_{3,Y}$, the phasors that represent the light at point Y from slits S_2 and S_3. Explain how you determined the phase differences between adjacent phasors.

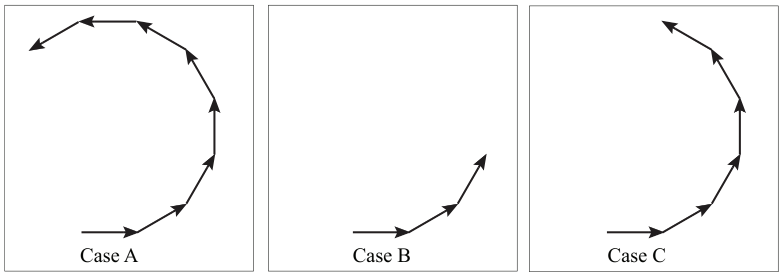
ii. What is the intensity at point Y when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for *three* slits. Explain your reasoning.

PHASORS

Name:____________________________
Section:

WO
HW–17

1. Coherent red light is incident on a mask with an unknown number of equally spaced narrow slits. Each of the phasor diagrams below corresponds to the same location on a distant screen. The only difference among the three physical situations is the number of slits.



a. How many slits are there in each case? Explain.

b. In case A, $\delta\varphi_{adj}$ is $\pi/6$. What is the corresponding value of δs_{adj} in terms of wavelength? Explain.

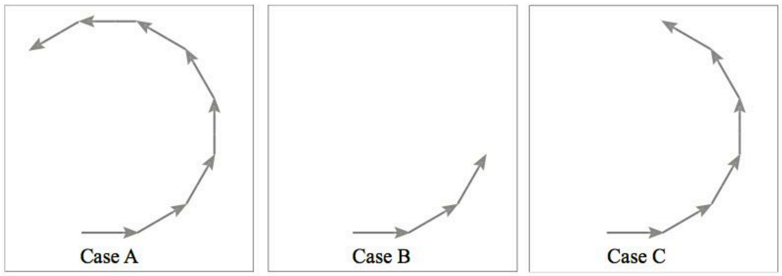
Would $\delta\varphi_{adj}$ and δs_{adj} be the same for cases B and C as it is for case A? Explain.

c. Rank the cases according to the intensity, from largest to smallest. Explain.

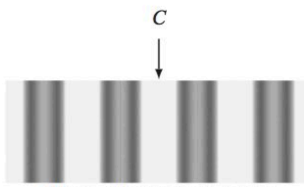
d. Based on your answers above, does the intensity always increase when the number of slits increases? Explain.

Figure B.9: *Phasors* tutorial homework administered in Spring 2018 to PHYS 123 (four pages).

e. Now suppose in each case one of the outermost slits is covered. In each case, would the intensity *increase, decrease,* or *stay the same?* Explain. As part of your explanation, sketch a possible phasor diagram for each case. The original phasor diagrams are duplicated for reference.

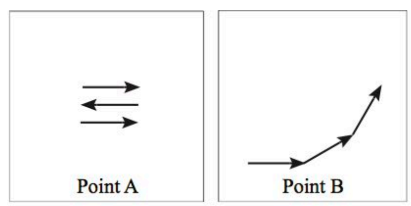


2. Coherent red light is incident on a mask with three equally spaced narrow slits. The interference pattern on the screen is shown at right. Recall that in this tutorial, the $\delta\varphi_{adj}$ of 3π and π are not equivalent.

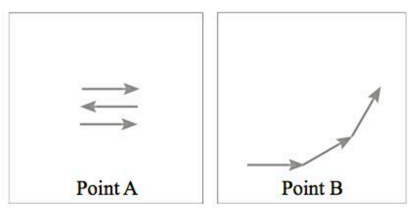


Three-slit pattern on screen

a. Consider the phasor diagrams below right corresponding to points A and B on the screen. The $\delta\varphi_{adj}$ for points A and B are 3π and $\pi/6$, respectively. Mark the locations of points A and B on the interference pattern and explain how you determined the locations.



b. Suppose a fourth slit is added to the right of the original three slits such that distances between adjacent slits are the same as before. In each case, would the intensity *increase, decrease,* or *stay the same?* Explain. As part of your explanation, sketch a possible phasor diagram for each case. The original phasor diagrams are duplicated for reference.



3. Consider each of the minima between the central principal maximum and the first maximum to the right of the center. Suppose that coherent red light were incident on a mask with four equally spaced narrow slits S_1, S_2, S_3, and S_4. Recall that there are three minima between adjacent principal maxima in this case.

   a. Suppose that each of the minima is represented by a phasor diagram.

   Predict whether all the phasor diagrams would look the same if the phasors were *not* labeled in any way. Explain.

   b. Consider each of the minima between the central principal maximum and the first maximum to the right of the center.

   i. What are the values of $\delta\varphi_{adj}$ for those minima?

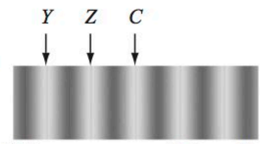
   ii. Use your answers above to sketch a phasor diagram for each minimum.

First minimum	*Second minimum*	*Third minimum*

   iii. Label the corresponding slits (S_1, S_2, S_3 or S_4) and $\delta\varphi_{adj}$ in each phasor diagram as was done in tutorial.

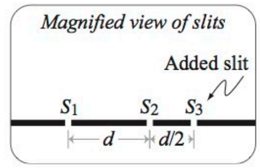
   The phasor diagrams you drew above should not all be identical. Check the value of $\delta\varphi_{adj}$ for each phasor diagram and resolve any inconsistencies.

4. The diagram at right illustrates the pattern that appears on a distant screen when coherent red light is incident on a mask with two identical, very narrow slits, S_1 and S_2. The two slits are separated by a distance d. Point C represents the center of the screen.



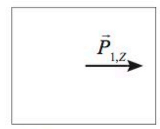
Pattern on screen with two slits

Suppose that a third slit, S_3, of equal width were added. It is a distance $d/2$ to the right of slit S_2 as shown in the figure below the pattern. (The new pattern on the screen is not shown.)



a. Consider point Z on the screen. Let phasor $\vec{P}_{1,Z}$ (see figure below right) represent the light at this point from slit S_1.

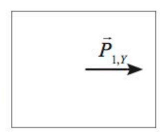
   i. In the figure, sketch $\vec{P}_{2,Z}$, the phasor that represents the light at point Z from slit S_2. Explain how you determined the phase difference between $\vec{P}_{2,Z}$ and $\vec{P}_{1,Z}$.



   ii. Sketch $\vec{P}_{3,Z}$, the phasor that represents the light at point Z from slit S_3. Explain how you determined the phase difference between $\vec{P}_{3,Z}$ and $\vec{P}_{2,Z}$.

   iii. What is the intensity at point Z when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for *three* slits. Explain your reasoning.

b. Now consider point Y on the screen. Let phasor $\vec{P}_{1,Y}$ (see figure at right) represent the light at point Y from slit S_1.



   i. In the figure, sketch $\vec{P}_{2,Y}$ and $\vec{P}_{3,Y}$, the phasors that represent the light at point Y from slits S_2 and S_3. Explain how you determined the phase differences between adjacent phasors.

   ii. What is the intensity at point Y when slit S_3 is added? Express your answer in terms of the intensity of a principal maximum for *three* slits. Explain your reasoning.

TWO-SOURCE INTERFERENCE

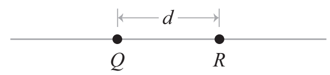
Name:____________________________
Section:

WO
HW–7

1. The top-view diagram at right illustrates two point sources, Q and R.

   a. On the diagram, indicate points for which the value of δs is (i) largest and (ii) smallest. (δs is the difference in distances from the point to the sources.) Indicate at least two points in each case and label the points (i) and (ii).

   b. What are the largest and smallest values of δs for this situation? Explain your reasoning.



2. Each of the diagrams at right shows all the nodal lines (dashed) and all the antinodal lines (solid) due to two point sources. The wavelength, λ, is the same in all three cases; the source separation is different. (The sources, which are not shown, lie along a horizontal line.)

   a. Label each nodal and antinodal line in the shaded region with the appropriate value of δs (in terms of λ).

   b. For each case, determine the source separation (in terms of λ). For any case(s) for which it is not possible to determine the source separation *exactly*, determine the source separation as closely as you can (*e.g.*, by giving the smallest range into which the source separation must fall). *Hint:* You may find it helpful to first rank the cases by source separation.

      Explain how you determined the source separations.

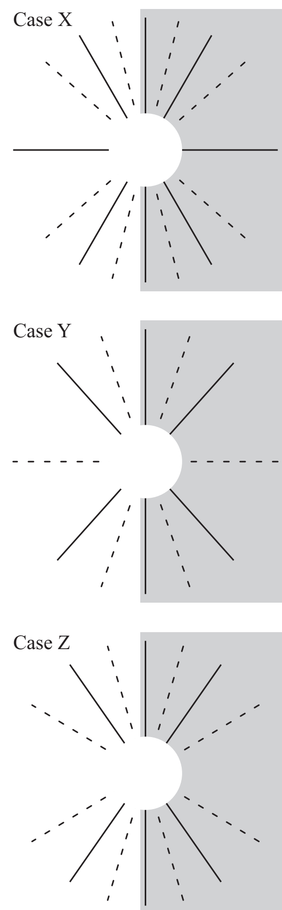
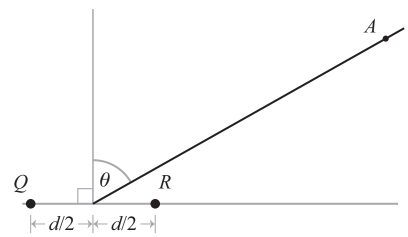
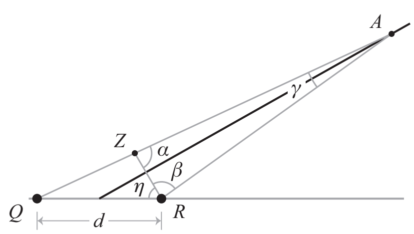


Figure B.10: *Two-source Interference* tutorial homework administered in Spring 2018 to PHYS 123 (five pages).

3. The diagram at right shows an arbitrary point, point *A*, that lies near two point sources of waves. In this problem, we consider how the path length difference to point *A* changes as point *A* is moved along the dark line shown, away from the sources.

   Point *Z* on the diagram below was chosen so that point *Z* and source *R* are equidistant from point *A*.





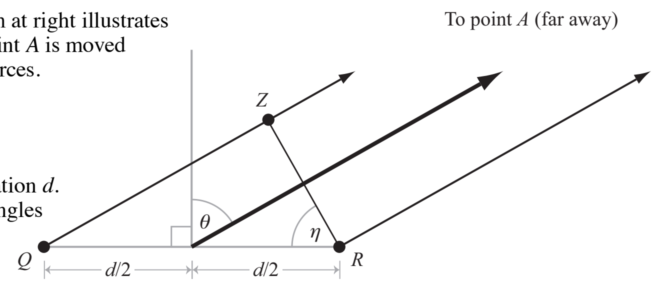
a. How do the angles α and β compare? Explain your reasoning.

b. Suppose that point *A* is moved away from the sources along the dark line. In the limit that point *A* is very far from the sources, what values do the angles α, β, and γ approach?

c. On the diagram above, indicate the line segment that represents how much farther point *A* is from *Q* than it is from *R*. Label this distance δs.

d. The enlarged diagram at right illustrates the limit in which point *A* is moved very far from the sources.

   In this limit, find an expression for δs in terms of the angle θ and the source separation *d*. (*Hint:* How do the angles η and θ compare?)



e. For what values of δs (in terms of λ) will there be:

   • maximum constructive interference (i.e., an antinode)?

   • complete destructive interference (*i.e.*, a node)?

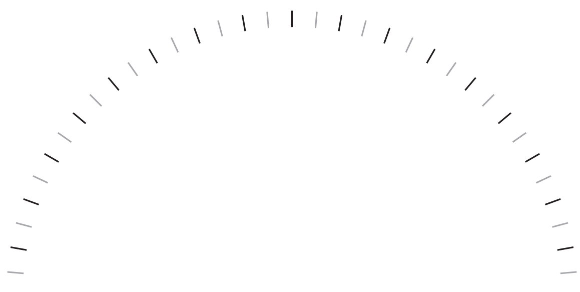
f. Use your answers from parts d and e to write equations that can be used to determine the angle(s) for which there will be:

- antinodal lines

- nodal lines

g. Determine the angles for which there will be nodal lines and antinodal lines for the case of two sources in phase, a distance 1.5λ apart.

i. Use your results to draw *accurate* nodal lines and antinodal lines on the diagram at right.



2 sources in phase
$d = 1.5\lambda$ between sources (not shown)

ii. Label each line from part i with the corresponding value of δs and θ.

iii. If the distance between the sources were increased, would the angle θ to the first nodal line *increase*, *decrease*, or *stay the same?* Explain your reasoning.

4. Consider the following *incorrect* statement referring to problem 3:

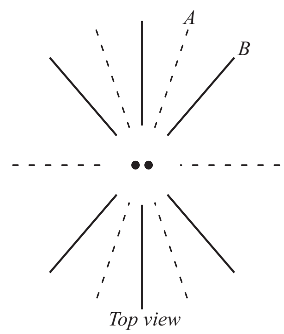
"As point A moves farther and farther away from the sources, the distances to the sources become more nearly equal, so the difference in distances is negligible. Thus the waves are more nearly in phase as point A moves farther and farther away from the sources."

What is the flaw in this argument? Explain your reasoning.

WO HW–10 *Two-source interference*

5. In this problem, you will be considering different pairs of point sources that produce period waves of wavelength λ in ripple tanks of water.

   a. Consider sources S_1 and S_2. All of the nodal lines (dashed) and antinodal lines (solid) produced by the two sources are as shown at right.

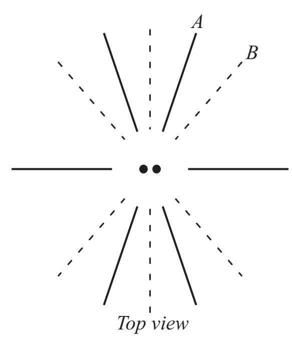
   

      i. Are the sources S_1 and S_2 in phase? Explain.

      ii. For each point along lines A and B, determine the path length difference δs and the phase difference $\delta\varphi$. Explain. In addition, indicate the units for each quantity.

δs_A	δs_B	units of δs
$\delta\varphi_A$	$\delta\varphi_B$	units of $\delta\varphi$

   b. Now consider sources S_3 and S_4. The interference pattern due to sources S_3 and S_4 is shown at right.

      i. Are the sources S_3 and S_4 in phase? Explain.

      ii. For each point along lines A and B, determine the path length difference δs and the phase difference $\delta\varphi$. Explain.

δs_A	δs_B
$\delta\varphi_A$	$\delta\varphi_B$

      iii. How do the values of δs and $\delta\varphi$ in parts a and b above compare? Specify which are the same and which are different.

c. Now instead consider sources S_5 and S_6, each producing waves of wavelength λ Suppose you know that the path length difference from sources S_5 and S_6 to a point, X, is λ. (The interference pattern is not shown.)

Is it possible to infer whether point X lies on a nodal line or an antinodal line? If not, what additional information would you need? Explain. (*Hint:* Check to be sure your answer is consistent with your answers in parts a and b.)

APPENDIX C

Unabbreviated Tasks

This appendix consists of unabbreviated versions of tasks discussed in this dissertation.

18. The particle is prepared at time $t = 0$ with wavefunction $\psi_a(x, 0) = C\{\phi_1(x) + \phi_2(x)\}$.

a. [4] Is it in a stationary state? Why (not)?

b. [6] Find C (take it to be real).

c. [6] Find $\langle E \rangle$.

d. [10] Find the probability of finding the particle in the left half of the well ($x < L/2$) at any time

19. [8] If the particle were prepared instead with wavefunction $\psi_b(x, 0) = C\{\phi_1(x) + i\phi_2(x)\}$, which of the answers to a-d in the previous problem would be different? 2 points for each. No need to evaluate anything or give explanation unless you want to.

Figure C.1: The unabbreviated version of Task 4.8 in chapter 4.

VITA

Tong Wan grew up in China. She received her Bachelor's degree in physics from Dalian University of Technology in 2011. She then came to the United States and joined the physics department at Washington State University as a doctoral student. After realizing her passion for physics education research, she received a Master's degree from Washington State University and began the Ph.D. program at the University of Washington in 2013. In 2018, she earned a Doctor of Philosophy in physics.